Spectra, Algorithms, and Data in Infinite Dimensions (L16)

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This is a new Part III course for 2026–27. It gives a coherent, example-driven 16-lecture introduction to modern spectral computation in infinite dimensions. Starting from first principles, the course develops the operator theory, computability framework, and error analysis needed to prove rigorous algorithms, impossibility results, and validated data-driven methods.

Spectra are one of the central organising objects of modern mathematics and its applications. They describe energy levels in quantum mechanics, stability in differential equations and control, resonances in waves, transport in materials, and coherent structures in dynamical systems. Increasingly, they also arise when one tries to infer dynamics from finite data. In finite dimensions, spectra are eigenvalues of matrices. In infinite dimensions the picture is richer and more delicate: spectra may be continuous, eigenvectors may not form a basis, and standard discretisations can produce convincing but wrong answers.

This course asks a foundational and practical question: what can algorithms reliably compute, verify, or rule out about spectra of infinite-dimensional operators? We begin with visual and concrete mathematical examples of spectral pollution, spectral invisibility, continuous spectrum, and severe ill-conditioning. These examples motivate a shift from computing eigenvalues of finite sections to algorithms based on the resolvent, pseudospectra, and smallest singular values. We then introduce the Solvability Complexity Index (SCI) as a language for proving positive algorithms, verification results, and impossibility theorems.

The course develops rigorous methods, with proofs of convergence and error control, including rectangular truncations and operator folding. It also treats spectral measures as the correct analogue of diagonalisation in the presence of continuous spectrum, with Stone's formula and smoothing kernels providing practical algorithms. The final part of the course explores modern applications, such as quasicrystals, partial differential equations, and data-driven Koopman spectral analysis of nonlinear dynamical systems. A concluding lecture discusses AI-assisted validated computation: as AI systems make plausible numerical claims cheap, proof objects, interval enclosures, and independent certificates become more central.

Indicative topics include:

- spectra beyond eigenvalues: continuous spectrum, essential spectrum, and spectral measures;
- spectral pollution, spectral invisibility, and ill-conditioning caused by naive discretisation;
- pseudospectra, the resolvent, injection moduli, and singular-value based spectral algorithms;
- the Solvability Complexity Index and the classification of computational spectral problems;
- verified computation of spectra and spectral measures, including rectangular truncations, operator folding, Stone's formula, and smoothing kernels;
- AI-assisted validated computation, interval enclosures, Krawczyk-type certificates, proof objects, and the changing standards for numerical claims;
- applications to PDEs, quasicrystals and aperiodic media, and data-driven Koopman spectral analysis.

Numerical pictures and model problems motivate the theory; the aim is to turn them into precise definitions, proofs, error estimates, and impossibility results explaining which computations can be trusted.

Prerequisites

The course is mathematically substantial, but the prerequisites are deliberately light: basic linear algebra and analysis, together with elementary normed and Hilbert space ideas. Functional analysis is helpful but not assumed; the spectral-theoretic notions needed for the course will be introduced as they arise. No prior numerical analysis or programming is required.

Preliminary Reading

Students who want optional preparation over the summer could review:

1. basic Hilbert space theory: bounded linear operators, adjoints, orthogonal projections, and compact operators;
2. the spectral theorem for compact self-adjoint operators;
3. elementary matrix singular values and pseudospectra.

A concise, self-contained set of preliminary notes will be provided before the course.

Literature

1. M. J. Colbrook, *Infinite-Dimensional Spectral Computations: Foundations, Algorithms, and Modern Applications*, Cambridge University Press, forthcoming.
2. E. B. Davies, *Linear Operators and their Spectra*, Cambridge University Press, 2007.
3. M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, 1980.
4. M. J. Colbrook, A. Horning and A. Townsend, "Computing spectral measures of self-adjoint operators", *SIAM Review* 63 (2021), no. 3, 489–524.
5. M. J. Colbrook and A. C. Hansen, "The foundations of spectral computations via the Solvability Complexity Index hierarchy", *Journal of the European Mathematical Society* 25 (2023), no. 12, 4639–4718.
6. M. J. Colbrook, "Ten Digits on a Train: AI-Assisted Verification of Two Eigenvalue Problems", arXiv:2606.23821, 2026.

Additional support

Exercise material will be embedded throughout the lecture notes, giving regular practice with proofs, computations, and interpretation. There will be three examples classes to discuss selected exercises and extensions, together with a one-hour revision class in the Easter Term.