Analysis of Partial Differential Equations

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This course serves as an introduction to the mathematical study of Partial Differential Equations (PDEs). The theory of PDEs is nowadays a huge area of active research, and it goes back to the very birth of mathematical analysis in the 18th and 19th centuries. The subject lies at the crossroads of physics and many areas of pure and applied mathematics.

The course will mostly focus on developing the theory and methods of the modern approach to PDE theory. Emphasis will be given to functional analytic techniques, relying on a priori estimates rather than explicit solutions. The course will primarily focus on approaches to linear, elliptic and evolutionary problems through weak formulations and energy estimates, with the prototypical examples being Laplace's equation, the heat equation, the transport equation, the wave equation and Schrödinger's equation.

The following topics will be studied.

- *Introduction and Cauchy–Kovalevskaya theorem.* Definition and examples of PDEs; well-posed problems; classification; from ODEs to PDEs; real analyticity; Cauchy–Kovalevskaya theorem for ODEs and (quasilinear) PDEs.
- *Spaces of functions.* Hölder spaces; weak derivatives; Sobolev spaces; approximation by mollifiers; extension and trace theorems; Sobolev embeddings.
- *Elliptic boundary value problems.* Weak formulation; solvability via Lax-Milgram; Fredholm alternative for elliptic PDEs and its consequences on the spectrum; regularity of solutions via difference quotient method.
- *Hyperbolic equations.* Notion of hyperbolicity for second order linear operators, weak formulation of initial boundary value problem; existence via Galerkin's method; uniqueness via vector field method; finite speed of propagation; first order hyperbolic PDEs and method of characteristics.

Prerequisites

There are no specific prerequisites beyond a standard undergraduate analysis background, in particular a familiarity with measure theory and integration. The course will be mostly self-contained and can be used as a first introductory course in PDEs for students wishing to continue with some specialised PDE Part III courses in the Lent and Easter terms.

Literature

As a preliminary reading and revision, students are encouraged to read through the online notes of one of the incarnations of the Part II Analysis of Functions course. Previous lecture notes for this Part III course are available at https://cmouhot.wordpress.com/teachings/ written by C. Mouhot, and at https://www.dpmms.cam.ac.uk/~zw253/teaching.html written by Z. Wyatt.

1. Evans, L. C., Partial Differential Equations, Springer, 2010.
2. John, F., Partial Differential Equations, Springer, 1991.

3. Brezis, H., Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, 2010.

4. Klainerman, S., Partial Differential Equations, Princeton Companion to Mathematics (editor T. Gowers), Princeton University Press, 2008.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.