An introduction to non linear analysis

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The 6th Clay problem has an elementary formulation: may incompressible fluids form singularities? This question and the description of singularities, at the heart of the mathematical study of non linear waves and their partial differential equations, is relevant in various areas of mathematical physics (astrophysics, non linear optics, electromagnetism ...) and has attracted an immense amount of mathematics in the last fourty years with inputs from very different fields: harmonic analysis, dynamical systems, variational methods.

This class is an introduction to the mathematical study of non linear waves. We will focus onto a canonical model, the non linear Schrödinger equation, and will also present some connections with both compressible and incompressible fluid dynamics. We will introduce in a self contained way using very little prerequisites the basic concepts that have emerged in the last fourty years, in particular soltions which are non linear bubbles of energy which propagate without deformation. The major concepts that we will address are: Cauchy theory and scattering à la Ginibre and Velo (1983), existence of solitons (using variational methods, dynamical systems or direct non linear bifurcation methods), stability of solitary waves (Cazenave-Lions 1983) and an introduction to singularity formation through self similar profiles and the minimal mass Theorem (Merle 1992).

Prerequisites

Basic notions of functional analysis (Hilbert and Banach spaces). Basis notion of distributions theory ($\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$). Basic notion of continuous Fourier transform (Plancherel).

Literature

1. R. Danchin; P. Raphaël, An introduction to non linear waves, https://www.dpmms.cam.ac.uk/professor-pierre-raphael-personal-homepage

2. J.-M. Bony: Intégration et analyse hilbertienne, *cours de l'École Polytechnique*, 2006.

3. J.-M. Bony : Cours d'analyse, théorie des distributions et analyse de Fourier, Éditions de l'École Polytechnique.

4. H. Brézis : *Analyse fonctionnelle. Théorie et applications*, Masson, 1984.

5. J. Ginibre et G. Velo, On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case, *Journal of Functional Analysis*, **32**, 1979, pages 1–32.

6. P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **1**, 1984, pages 109–145.

7. F. Merle, Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power, *Duke Math. Journal*, **69** (2), 1993, pages 427–454.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.