Topics in Infinite Groups(L16)

Non-Examinable (Graduate Level)

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The central theme of this course will be the Sigma (or BNS) invariants of finitely generated groups. These were originally introduced by Bieri and Strebel in 1980 as a means of characterising which metabelian groups are finitely presented. (A group is metabelian if it maps onto an abelian group with abelian kernel.) More generally the invariants give information about the kernels of maps onto free abelian groups and whether they have good homological/homotopical finiteness properties. About the same time it was shown by Bieri and Groves that the invariants in this context are rational polytopes.

Later in the 1980s, working with Walter Neumann, the definitions were extended to cover all finitely generated groups. At the same time Brown showed that the invariant records the existence of certain types of tree actions of the group.

Since the 1980s the BNS invariants have been generalised and calculated for a huge variety of groups, and they have been shown to have connections with a lot of interesting mathematics. For example, in the case of fundamental groups of 3-manifolds there is a link with the Thurston norm on the homology. There is also a link with tropical geometry and the compactifications of closed subvarieties of the complex n-torus via toric varieties. Also, another version of the definition concerns Novikov homology, and when it vanishes.

Just as in the original case the invariants are often shown to be rational polytopes. But this is not always so, for example in the case of some of the Thompson groups of piecewise linear homeomorphisms of the real line, which provide such good counterexamples to conjectures.

The aim of the course is to present the original work and then visit some of the mathematics involved in calculating various examples from geometric group theory.

Prerequisites

I shall try to keep prerequisites to a minimum. For the original work it will be useful to know some basic commutative algebra (about valuations).

Literature

1. Bieri, R., Groves, J.R.J., *The geometry of the set of characters induced by valuations*, J. reine Angew. Math. **347**, 168-195 (1984).

2. Bieri, R., Neumann W., Strebel R., *A geometric invariant of discrete groups*, Invent. Math. **90**, 451-477 (1987).

3. Bieri, R., Strebel R, *Valuations and metabelian groups*, Proc. London Math. Soc. **41**, 439-464 (1980).

4. Brown K.S., *Trees, valuations, and the Bieri-Neumann-Strebel invariant*, Invent. Math. **90**, 479-504 (1987).

5. Kielak, D., *The Bieri-Neumann-Strebel invariants via Newton polytopes*, Invent. Math. **219**, 1009-1068 (2020).

6. Sociu, A.I., *Sigma invariants and tropical varieties*, Math. Ann.**380**,1427-1463 (2021).