Modular Representations of Finite Groups (L16)

Non-Examinable (Graduate Level)

Stuart Martin

Modular representation theory is the representation theory of finite groups over a field of prime characteristic p. It is very different from the characteristic zero theory which most undergraduates are exposed to in a first course, though there is a rich interplay between the two. If G is a finite group of Lie type such as $\mathrm{GL}_n(\mathbf{F}_q)$ then the normaliser of a Sylow subgroup is a Borel subgroup, and more generally parabolic subgroups are normalisers of certain p-subgroups. Since the parabolic subgroups play such an important role in other areas of representation theory (including the theory of automorphic forms) it is not surprising that the normalisers of p-groups should play an important role in the representation theory of arbitrary finite groups. What is surprising is the Brauer correspondence, a deep connection between the representation theory of G and normalisers of p-subgroups, an altogether new phenomenon in the modular theory. More recently, beginning with work of Lascoux, Leclerc and Thibon, deep connections have been found with the representation theory of quantum groups and modular representation theory of (for example) of the symmetric group.

Taking a step back, the first really major developments came with Richard Brauer, who from 1935 onwards exploited this rich and virtually untapped area of mathematics. Brauer's methods were mainly character-theoretic. What is now called the theory of Brauer characters determines the composition factors of the representation but, unlike the situation over $\mathbf{C}$, not the equivalence type. One of Brauer's main goals was finding numerical constraints on the orders and internal structure of finite simple groups, and his methods were later used by Glauberman and others in the Classification program for finite simple groups.

The next revolution in the theory came with Sandy Green, who considered the modules themselves. His techniques were completely different to those of Brauer, and his main goals lay in understanding the modules ring-theoretically, rather than Brauer who worked mostly with characters and their blocks (a set of equivalence classes of characters). The aim of this course is to give a flavour of both classical techniques of Brauer and the more modern techniques of Green. We'll discuss Brauer characters, defect groups, blocks, decomposition numbers as well as projective and injective modules (as time allows).

I'll aim to cover some of the following:

- Review of assumed and basic material: group algebras, representations and modules, reducibility and decomposability, Maschke's theorem, tenors and homs, Frobenius reciprocity, Mackey's theorem, maximal and primitive ideals, Jacobson radical, complete reducibility, semisimple rings, Artin-Wedderburn structure theorem.
- Modular character theory: p-singular and p-regular elements, Brauer characters, Grothendieck groups.
- Irreducible, projective and injective modules: DVRs, (splitting) p-modular systems, forms, decomposition numbers and the decomposition matrix, counting modular irreducible modules.
- Projective indecomposable modules and their characters: the Higman criterion, primitive orthogonal idempotents, idempotent refinement.
- Block theory: the socle and the head, Cartan matrix, lifting projectives, central idempotents.

- Central characters: the central character of an ordinary irreducible mod p determines the block.
- (If time permits) Further block theory: defect groups, relative projectivity and blocks of defect 0. (Statement of) the Alperin conjecture, some comments on local–global conjectures.

Prerequisites

Basic group theory; ordinary representations/character theory from the Part II (or equivalent) course; Sylow theory for finite groups; commutative algebra from the relevant Part III course (rings, ideals, completions, local rings, primality, maximality, the Artin-Wedderburn theorem); some categorical nonsense. The Part III course on the representations of complex semisimple Lie algebras is complementary to this course.

Preliminary Reading

1. C.W. Curtis, Pioneers of representation theory: Frobenius, Burnside, Schur and Brauer (AMS 1999)
2. P. Etingof *et al*, Introduction to representation theory (AMS 2011)
3. I.M. Isaacs, Character theory of finite groups (Dover reprint 1994)
4. G.D. James & M.W. Liebeck, Representations and characters of groups (CUP, 2nd edn 2001)
5. W. Ledermann, Introduction to group characters (CUP, 2nd edn, 1987)

Literature

1. J.L. Alperin, Local representation theory (CUP 1986)
2. D.J. Benson, Representations and cohomology, Vol 1 (CUP 1991)
3. D.A. Craven, Representation theory of finite groups: a guidebook (Springer 2019) - it has 574 references!
4. P. Schneider, Modular representation theory of finite groups, (Springer 2013)
5. P. J. Webb, A course in finite group representation theory (CUP 2016)