Introduction to Geometric Representation Theory (L24)

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This is a course about algebra and representation theory. Given a group, for example SL_2, or perhaps a Lie algebra, such as sl_2, one would like to understand how it acts in the world — to describe each of its representations. We want to know how many there are, what their dimension is as a vector space, how they break up into representations of natural subgroups or algebras – for a representation of a finite group that means determining its character – and so on.

Sometimes one can do this just with linear algebra. You will have seen this in the Lie algebra course, where the finite dimensional representations of a complex semisimple Lie algebra are completely described.

Most of the time this is not possible. For example, you might want to understand how a Verma module, which is an 'induced' representation of a simple Lie algebra, decomposes into irreducibles. Strenuous computations in small examples such as sl_4 produce answers that appear quite mysterious.

About 45 years ago it was discovered (the 'Kazhdan-Lusztig conjecture') that these mysterious answers are in fact topological invariants — they are determined by the cohomology of natural algebraic varieties associated to the representation theory. This was proved by Beilinson and Bernstein, using D-modules, perverse sheaves and the Weil conjectures.

Since then it has been understood that for most interesting algebras or groups – even finite groups like $SL_n(F_q)$ – that certain categories of sheaves on algebraic varieties control their representation theory.

The two widely used names for this area of study are 'geometric representation theory', and 'the Geometric Langlands programme'.

This is intended to be a *gentle* Part III introductory course on geometric representation theory. I'll assume you know some very basic algebraic geometry, and not much else. The Lie algebra course will be helpful though not entirely necessary, as most of the theorems and constructions are already interesting for sl_2. Hopefully the course will motivate you to pick up any technical background you need but do not have.

We will begin with the Lie algebra representations of sl_2 and the Beilinson-Bernstein theorem, by hand (this will be really good revision for the part III algebraic geometry course!) then continue with a few other topics.

These may include 1) the Borel-Weil-Bott theorem and introduction to the geometry of the flag variety, 2) the Beilinson-Bernstein theorem describing representations of semisimple complex Lie algebras, 3) a brief introduction to complex representations of Chevalley groups, such as $SL_2(F_q)$ and Deligne-Lusztig theory, 4) the mod p representations of these same groups, and 5) Jantzen's theorems on reduction mod p of the complex representations.

In each case we will do SL_2 by hand, with very little technical background required, and then only state and prove some indicative theorems about the general case (as this is an introductory course).

If time remains we will continue with more recent and harder subjects, which will not be examinable. (When lectures are over the course moodle will list the examinable topics).

Prerequisites

For the first two topics I'll assume you know what a Lie algebra is and have some basic algebraic geometry, but not a lot else. For the next few topics, you need to know what a character of a representation is (the Part II representation theory class is more than sufficient).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.