Amenability

Non-Examinable (Graduate Level)

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Amenability was originally introduced by von Neumann in an attempt to explain the Hausdorff–Banach–Tarski paradox. Since then, it has become the cornerstone of analytic group theory, and one of the most important topics in the theory of infinite discrete groups in general. The goal of the course is to introduce you to this topic, and to the techniques of analytic group theory along the way.

One of the most fascinating things about amenability is that it has a shocking number of equivalent characterisations. These involve: invariant measures, paradoxical decompositions, fixed point theorems, almost invariant sets, almost invariant vectors, harmonic functions, operator algebras, bounded cohomology, symbolic dynamics... The core of the course will consist in proving some of these equivalent characterisations.

Additional topics will be covered depending on the interests of the audience. These will be chosen among the following.

- Counterexamples to the von Neumann–Day problem.
- Quasi-isometries, orbit equivalence, and the Ornstein–Weiss Theorem.
- Relation to dynamics, groups acting on the circle and the Margulis alternative.
- Amenability over non-Archimedean fields.
- Gromov's polynomial growth Theorem.
- Antipodes of amenability: property (T).

Prerequisites

A first course in group theory and a first course in topology (beyond metric spaces) will be assumed throughout.

Some basics of functional analysis are fundamental to the study of amenability. Specifically: normed vector spaces, the weak-$*$ topology, the Banach–Alaoglu Theorem and the Hahn–Banach Theorem, possibly nets and subnets. Since not everyone has done a functional analysis course in their undergrad, I will quickly review these notions and results when they are needed, but you are encouraged to catch up on them on your own before the course starts. These are all basic enough that wikipedia will do, otherwise any functional analysis textbook (Rudin / Conway / Zimmer) will have all you need.

Literature

I will write lecture notes throughout the semester and make them available week by week. Here are some general references on amenability, where you will be able to find everything that the course will cover and more.

1. T. Checcherini-Silberstein and M. Coornaert, *Cellular automata and groups.* Springer, 2010.

2. K. Juschenko, *Amenability of discrete groups by examples.* AMS, 2022.

3. T. Cohen, T. Gelander. *An invitation to analytic group theory.* arXiv, 2024.

4. V. Runde. *Lectures on amenability.* Springer, 2002.

Additional support

Some steps in the course will be left as exercises. These will not have accompanying solutions or be graded, but you are encouraged to give them a try and there will be dedicated office hours where we can talk about them.