UNIVERSITY OF CAMBRIDGE

# M. Phil. in STATISTICAL SCIENCE

Monday, 8 June, 2009 9:00 am to 11:00 am

## OPTIMAL INVESTMENT

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Baron Kiwi, the shoe-polish billionaire, appoints an investment manager to handle his investments for him. His initial wealth is $w_0$, and the manager can invest in a riskless asset returning a constant interest rate $r$, and in $d$ risky assets, whose prices $S_t \equiv (S_t^1, \ldots, S_t^d)^T$ evolve as log-Brownian motions:

$$dS_t^i = S_t^i(\sum_{j=1}^{d} \sigma_{ij} dW_t^j + \mu_i dt) \qquad (i = 1, \ldots, d),$$

where the $W^j$ are independent standard Brownian motions. The wealth process generated by the manager satisfies

$$dw_t = rw_t dt + \sum_i w_t \pi_t^i (\sum_j \sigma_{ij} dW_t^j + (\mu_i - r)dt),$$

where $\pi_t^i$ denotes the fraction of wealth invested in asset $i$ at time $t$. Baron Kiwi wants the investment manager to increase his wealth by time $T$, when his daughter is due to buy a University. If the manager's (admissible) investment results in a wealth $w_T$ at time $T$, then the Baron has promised to pay the manager

$$y_T \equiv aw_T \exp(-\tfrac{1}{2}\varepsilon \int_0^T |\sigma^T \pi_s|^2 \, ds),$$

where $a > 0$, $\varepsilon > 0$, and the Baron's intention in reducing the payment by the exponential factor is to make the manager less keen to follow risky strategies.

If the manager's objective is to maximise $EU(y_T)$, where

$$U(x) = \tfrac{x^{1-R}}{1-R}$$

for some positive $R \neq 1$, prove that his optimal policy is exactly that which would be followed by an agent with utility

$$U_0(x) = \tfrac{x^{1-R-\varepsilon}}{1-R-\varepsilon}$$

and objective to maximize $EU_0(w_T)$.

Determine this optimal policy explicitly.

**2** An agent may invest in a single risky asset whose price process $S$ evolves as

$$dS_t = S_t(\sigma dW_t + \mu dt)$$

and in a riskless bank account. The interest paid on the bank account is at rate $r + \varepsilon > r$ until a random time $\tau$ which has an exponential distribution with mean $\lambda^{-1}$, and after that time the interest rate falls to $r$ and stays there. The agent's objective is to maximise

$$E \int_0^\infty e^{-\rho t} U(c_t)\, dt,$$

where $c_t$ is the rate of consumption withdrawal, $\rho > 0$ is constant, and $U(x) = x^{1-R}/(1-R)$ for some $R > 0$ different from 1.

(i) Consider first the case where the interest rate has already changed. Write down the agent's value function and optimal investment/consumption policy. [*You may use without derivation any standard results from the course.*]

(ii) Now consider the case where the interest rate has not yet changed. Obtain an equation which characterises the agent's value function and optimal investment/consumption policy, and find the value function as explicitly as you can.

**3** (i) Suppose that an agent may invest in a riskless bank account yielding interest at constant rate $r$, and in a stock, whose price $S_t$ at time $t$ evolves as

$$dS_t = S_t(\sigma dW_t + \mu dt).$$

If the agent aims to maximise the objective

$$(*) \qquad E\int_0^\infty e^{-\rho t}U(c_t)\,dt,$$

where $U'(x) = x^{-R}$ for some positive $R \neq 1$, and $c_t$ is rate of consumption withdrawal, write down the dynamics for the agent's wealth, and by finding the Hamilton-Jacobi-Bellman equations for this problem, or otherwise, show that the agent's optimal policy is to invest a fixed proportion $\pi_M$ of his wealth in the risky asset, and to consume at a rate $c_t = \gamma_M w_t$ for some constant $\gamma_M$, where you should identify $\pi_M$ and $\gamma_M$ explicitly in terms of the parameters of the problem.

[*You should assume that* $\rho + (R-1)(r + (\mu - r)^2/2\sigma^2 R) > 0$. *You are not required to give a verification proof.* ]

(ii) Suppose that there is a single productive asset in the world, which generates consumption goods at rate $\delta_t$, where for some constants $\sigma > 0$, $\alpha$,

$$d\delta_t = \delta_t(\sigma dW_t + \alpha dt).$$

There is a market in shares in the productive asset, and in riskless borrowing/lending of the consumption good.

Consider a representative-agent economy, where the single agent has objective (*). Show that if the share price process $(S_t)_{t\geqslant 0}$ is

$$S_t = \frac{\delta_t}{\rho + (R-1)(\alpha - \sigma^2 R/2)}$$

and the riskless rate of interest is constant, equal to

$$r^* = \rho + \alpha R - \tfrac{1}{2}(1+R)\sigma^2 R$$

then the markets clear; the representative agent faced with these prices will at all times hold one share in the productive asset, and will put no wealth into riskless borrowing/lending. Show also that the optimal consumption process is $\delta_t$.

[*You should assume that* $\rho + (R-1)(\alpha - \sigma^2 R/2) > 0$. ]

**4** (i) Suppose that $X_t = W_t + \alpha t$, where $W$ is a standard Brownian motion, and let $\mathcal{X}_t \equiv \sigma(X_u : 0 \leqslant u \leqslant t)$. The drift $\alpha$ is constant but unknown; it takes one of the values $a$ or $-a$, where $a > 0$. Each of the two possible values has equal prior probability. Mr Bayes observes the process $X$, and makes inference on the unknown drift $\alpha$. Show that after observing until time $t$ his posterior for $\alpha$ takes the form

$$P[\, \alpha = a \mid \mathcal{X}_t \,] = \frac{e^{aX_t}}{e^{aX_t} + e^{-aX_t}}.$$

Deduce that the dynamics of $X$ in the filtration $(\mathcal{X}_t)$ can be expressed as

$$dX_t = d\hat{W}_t + a \tanh aX_t \, dt,$$

where $\hat{W}$ is a $(\mathcal{X}_t)$-Brownian motion.

(ii) Mr Bayes wishes to invest in a market where there is a riskless bank account yielding interest at constant rate $r$, and a single risky asset whose price $S_t$ at time $t$ is given as

$$S_t = S_0 \exp(\sigma X_t + (\mu - \tfrac{1}{2}\sigma^2)t),$$

where $X$ is as in part (i). His objective is to maximize $E \int_0^\infty e^{-\rho t} U(c_t)\, dt$, where $\rho > 0$ is constant, $c_t$ is his rate of consumption withdrawal, and $U'(x) = x^{-R}$ for some positive $R \neq 1$.

Derive the Hamilton-Jacobi-Bellman equation for Mr Bayes' value function, and obtain an expression for the optimal investment in the risky asset.

Assume now that $\mu > r$, and $0 < R < 1$. Another agent, Mr Smart, knows that in fact $\alpha = a$; his objective is the same as Mr Bayes'. By considering Mr Smart's value function $V$, and the process

$$Y_t = \int_0^t e^{-\rho t} U(c_t)\, dt + e^{-\rho t} V(w_t),$$

where $c$ is Mr Bayes' optimal consumption process, and $w$ is Mr Bayes' optimal wealth process, show that $Y$ is a supermartingale.

Deduce that Mr Bayes' value cannot exceed Mr Smart's value.

## END OF PAPER