UNIVERSITY OF CAMBRIDGE

M. PHIL. IN STATISTICAL SCIENCE

Friday, 29 May, 2009 1:30 pm to 4:30 pm

MATHEMATICS OF OPERATIONAL RESEARCH

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Mathematics of Operational Research

Given vectors b, c and $m \times n$ matrix $A = (a_{ij})$, and $S \subseteq \{1, \ldots, m\}$ define $P(S)$ as the linear program

$$\text{maximize } c^\top x\,,$$

$$\text{such that } x \geqslant 0, \quad \text{and} \quad \sum_{j=1}^{n} a_{ij}x_j \leqslant b_i\,, \quad \text{for all } i \in S\,.$$

It is desired to find the optimal value of $P(\{1, \ldots, k\})$ for all $k \in \{1, 2, \ldots, m\}$ for which there exists a feasible solution. Starting with $P(\{1\})$, and then proceeding from its solution, use the dual simplex algorithm to solve this problem for the data

$$c^\top = (1, 1, 1, 1)\,, \quad A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ -1 & -1 & -4 & -5 \\ 0 & 1 & 2 & 2 \\ 4 & 2 & 0 & 6 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 11 \\ -6 \\ 1 \\ 20 \end{pmatrix}.$$

Consider the problem, having input data of arbitrary b, c, A and k, as follows:

Does there there exist a set S of size k such that $P(S)$ is feasible?

Explain why this problem is likely to be $\mathcal{NP}$-complete.

2 Mathematics of Operational Research

Suppose we are given a graph $G = (V, E)$, having n vertices and m edges. We are also given a set of edge weights $\{c_e,\ e \in E\}$, and a number k, all of these being integers in the range 1 to 100. For fixed vertices s and t, let Π be the set of all paths from vertex s to vertex t. A set of edge numbers $\{x_e,\ e \in E\}$ is said to be feasible if

$$\sum_{e \in E} c_e x_e \leqslant k,$$

$$\sum_{e \in p} x_e \geqslant 1\,, \quad \text{for all } p \in \Pi,$$

$$0 \leqslant x_e \leqslant 1\,, \quad \text{for all } e \in E.$$

It is desired to determine whether or not the feasible set, P, is not empty.

Show that the number of constraints in a general instance of this problem is not bounded by any polynomial function of n and m.

Show that if P is not empty, then it must be contained in a m dimensional sphere of volume no more than $O(200^m)$.

Explain how you could use the ellipsoid algorithm to solve this problem, so that the worst-case running time is bounded by a polynomial in n. You may state without proof facts about the algorithm.

Explain how you will solve the problem of checking (in polynomial time) whether or not the point z_t at the centre of an ellipsoid $E(z_t, D_t)$ satisfies the constraints.

3 Mathematics of Operational Research

Suppose that n facilities are to be placed at n locations, with one facility per location. A feasible solution can be associated with π, a permutation of $I = \{1, \dots, n\}$, which dictates that facility i be assigned to location $\pi(i)$. In the quadratic assignment problem (QAP) the data are matrices $A = (a_{i,j})$ and $B = (b_{i,j})$, and we wish to find

$$\mathrm{OPT} = \min_{\pi} f(\pi)\,,$$

where

$$f(\pi) = \sum_i \sum_{j\,:\,j \neq i} a_{i,j} b_{\pi(i),\pi(j)}\,.$$

Assuming that the travelling salesman problem is $\mathcal{NP}$-complete, show that QAP is $\mathcal{NP}$-complete.

Define

$$\ell_{i,k} = \min_{\pi} \sum_{j\,:\,j \neq i} a_{i,j} b_{k,\pi(j)}$$

where π is a one-to-one mapping of $I - \{i\}$ to $I - \{k\}$. Let Π_k be the subset of permutations of I in which $\pi(1) = k$. Define

$$g(\Pi_k) = \min_{\pi \in \Pi_k} \sum_i \ell_{i,\pi(i)}\,.$$

Explain why $g(\Pi_k)$ is a lower bound on $\min_{\pi \in \Pi_k} f(\pi)$.

What algorithm could you use to find $g(\Pi_k)$?

In a QAP with $n = 3$, suppose A, B, and $L = (\ell_{i,j})$ are

$$A = \begin{pmatrix} \cdot & 2 & 7 \\ 2 & \cdot & 4 \\ 7 & 4 & \cdot \end{pmatrix}, \quad B = \begin{pmatrix} \cdot & 5 & 3 \\ 5 & \cdot & 4 \\ 3 & 4 & \cdot \end{pmatrix}, \quad L = \begin{pmatrix} 31 & 38 & 29 \\ 22 & 26 & 20 \\ 41 & 48 & 37 \end{pmatrix}.$$

Use a branch and bound algorithm to find OPT. You should initially partition the solution space into three sets, Π_1, Π_2 and Π_3.

4 Mathematics of Operational Research

(a) Describe three heuristic methods for finding good solutions to $\mathcal{NP}$-hard problems. Say how these might be applied to the following problem.

'Winner Determination Problem' (WDP): A set of bidders, $M = \{1, \ldots, m\}$, is bidding for a set of items, $N = \{1, \ldots, n\}$. For each subset $S \subseteq N$, bidder i makes a nonnegative bid, say $v_i(S)$. Having received all bids, the auctioneer wishes to partition the items into disjoint subsets, $S_1, \ldots, S_m$, which he can assign to bidders $1, \ldots, m$ respectively, to obtain

$$\mathrm{Opt}(v) = \max_{S_1,\ldots,S_m} \sum_{i \in M} v_i(S_i)\,.$$

(b) Now suppose that all v_i are increasing and submodular. This means that for all j, S and T with $j \notin S$ and $S \subseteq T \subseteq N$,

$$0 \leqslant v_i(T + \{j\}) - v_i(T) \leqslant v_i(S + \{j\}) - v_i(S)\,.$$

The following heuristic algorithm is proposed for WDP.

0. Set $S_i = \emptyset$ for all $i \in M$, and $S_0 = N$.
1. Find $i \in M$ and $j \in S_0$ such that $v_i(S_i + \{j\}) - v_i(S_i)$ is maximal. Let $S_i := S_i + \{j\}$ and $S_0 := S_0 - \{j\}$.
2. Repeat step 1 until $S_0 = \emptyset$.
3. Return the solution $S_1, \ldots, S_m$, and $A(v) = \sum_i v_i(S_i)$.

Use induction on n to prove that this is a polynomial time approximation algorithm such that $A(v) \geqslant \frac{1}{2}\,\mathrm{Opt}(v)$.

Hint. Without loss of generality, suppose that the algorithm begins by allocating item n to player m. Consider a new problem in which items $\{1, \ldots, n-1\}$ are to be allocated, with bids

$$v_i'(S) = v_i(S)\,, \quad i \in \{1, \ldots, m-1\}$$
$$v_m'(S) = v_m(S + \{n\}) - v_m(\{n\})$$

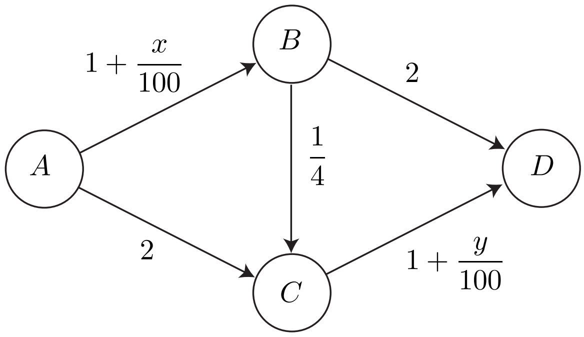
You may assume that v_i submodular implies v_i' submodular. Start by showing that $A(v) = v_n(\{m\}) + A(v')$. Then, by considering an allocation that achieves $\mathrm{Opt}(v)$ and modifying it by reallocating item n to bidder m (if it is not already so allocated), show that $\mathrm{Opt}(v') \geqslant \mathrm{Opt}(v) - 2v_m(\{n\})$.

5 Mathematics of Operational Research

Explain what is meant by a Nash equilibrium in a n person non-zero sum game.

State conditions under which a Nash equilibrium is guaranteed to exist.

In the road network below each of n players wishes to choose a route from A to D. Each player experiences a delay that is the sum of the delays on the links of his route. There is a delay of $1 + x/100$ on link AB when x players use that link, and a delay of $1+y/100$ on link CD when y players use that link. The delays on links AC, BD and BC are 2, 2, and 1/4.



Give, in its simplest form, a set of necessary and sufficient conditions for there to be a Nash equilibrium in which n_1, n_2 and n_3 players travel on routes ABD, ACD and $ABCD$ respectively. Hence show that when $n = 100$ there is a equilibrium at $n_1 = n_2 = 25$, $n_3 = 50$.

Is this the only equilibrium in pure strategies?

Show that it would be possible for the players to follow routes that make them all better off, but that this is not a Nash equilibrium.

Find a symmetric equilibrium, i.e., one in which all players use the same strategy.

6 Mathematics of Operational Research

Explain what is meant by the characteristic function, v, of a coalitional game with a set of players $N = \{1, \ldots, n\}$.

Describe the Shapley value and say what makes it an attractive solution concept.

A game is said to be convex if its characteristic function satisfies

$$v(S + \{i\}) - v(S) \leqslant v(T + \{i\}) - v(T)\,, \quad \text{for all } S \subset T \subseteq N \text{ and } i \notin T\,.$$

Suppose (v, N) is convex and let $\phi(v, N) = (\phi_1(v, N), \ldots, \phi_n(v, N))$ be the vector of its Shapley values. Consider the game (v, T) where $T \subset N$. This is the game in which only players in subset T participate. Show that $\phi_i(v, N) \geqslant \phi_i(v, T)$ for all i.

Hence show that $\phi(v, N)$ lies in the core of the game (v, N).

A firm consists of an entrepreneur and his workers. The firm cannot operate without the entrepreneur. For any nonnegative integer k, the entrepreneur and k workers can produces profit $p(k)$ that can be shared amongst them. Assuming the firm has n workers, use the Shapley value to calculate an expression for the 'fair' wage of a worker.

What is this fair wage when $p(k) = \alpha k$?

Prove that if $p(k)$ is a convex nondecreasing function of k then the Shapley value lies within the core of this game.

END OF PAPER