UNIVERSITY OF CAMBRIDGE

M. Phil. in STATISTICAL SCIENCE

Monday, 8 June, 2009   1:30 pm to 4:30 pm

STOCHASTIC CALCULUS AND APPLICATIONS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ be a filtered probability space. Define the *previsible* σ-*algebra* $\mathcal{P}$ and explain what is meant by a *simple process.* (We let $\mathcal{S}$ be the space of simple processes). Give the definition of the stochastic integral $H \cdot M$ of a simple process $(H_s, s \geqslant 0)$ with respect to a continuous martingale M which is bounded in L^2. (We let $\mathcal{M}_c^2$ be the space of continuous martingales bounded in L^2). Give the definition of the quadratic variation $[M]$ of a continuous local martingale M and explain how you can compute it from the path $(M_t, t \geqslant 0)$ using an approximation procedure. [You are not required to prove the existence of the quadratic variation or to justify your approximation procedure.]

(b) Let $H \in \mathcal{S}$ and $M \in \mathcal{M}_c^2$. Show that $H \cdot M \in \mathcal{M}_c^2$ and that

$$\mathbb{E}\left((H \cdot M)_\infty^2\right) \leqslant \|H\|_\infty^2 \mathbb{E}((M_\infty - M_0)^2).$$

(c) Show that for $H \in \mathcal{S}$ and $M \in \mathcal{M}_c^2$, we have in fact the equality

$$\mathbb{E}\left((H \cdot M)_\infty^2\right) = \mathbb{E}\left(\int_0^\infty H_s^2 d[M]_s\right).$$

Deduce that $[H \cdot M] = H^2 \cdot [M]$. [*Hint:*You may use the Optional Stopping Theorem provided that you state it clearly.]

2 Let M be a continuous local martingale and let A be a finite variation process such that for some nonrandom constant $K < \infty$ we have:

$$\sup_{s \geqslant 0}(|M_s| + V_s) \leqslant K \tag{1}$$

where V_s denote the total variation of A at time s. Let $X_t = M_t + A_t$, and assume that $M_0 = A_0 = 0$.

(a) Let ϕ be a nondecreasing function of class $\mathcal{C}^1$ such that $\phi(x) = -1$ for $x \leqslant 0$ and $\phi(x) = 1$ for $x \geqslant 1$. For all $n \geqslant 1$, define a function f_n such that $f_n(0) = 0$ and for all $x \in \mathbb{R}$, $f_n'(x) = \phi(nx)$. Show that $f_n(X_t)$ is a continuous semimartingale and give its Doob-Meyer decomposition. Show that as $n \to \infty$,

$$\int_0^t f_n'(X_s)dM_s \longrightarrow \int_0^t \operatorname{sgn}(X_s)dM_s$$

in the u.c.p. sense (uniformly on compacts in probability), where $\operatorname{sgn}(x) = 1_{\{x>0\}} - 1_{\{x \leqslant 0\}}$ is the (left-continuous) function which gives the sign of x. [*Hint*: Apply Itô's isometry property to the difference of those two integrals.]

(b) Show that as $n \to \infty$,

$$\int_0^t f_n'(X_s)dA_s \longrightarrow \int_0^t \operatorname{sgn}(X_s)dA_s$$

almost surely for all $t \geqslant 0$ simultaneously.

(c) Deduce from (a) and (b) that if $Z_t = |X_t| - \int_0^t \operatorname{sgn}(X_s)dX_s$, then Z is a nondecreasing process almost surely. Conclude that $|X|$ is a continuous semimartingale. Show that the result remains true if we no longer assume (1), i.e., if X is any continuous semimartingale starting at 0.

3 (a) State the Dubins-Schwartz theorem. Let (M^1, M^2) be two continuous local martingales in a common filtration $(\mathcal{F}_t, t \geqslant 0)$, such that $[M^1, M^2]_t = 0$ and $[M^1]_t = [M^2]_t$, almost surely for all $t \geqslant 0$. (Here $[M, N]$ denotes the covariation of the two continuous local martingales M and N, and $[M]$ stands for $[M, M]$). We also assume that if $A_t = [M^1]_t$, then $A_\infty = \infty$ almost surely. Show that if $\tau_r = \inf\{t \geqslant 0 : A_t \geqslant r\}$, then

$$B_r = (M^1_{\tau_r}, M^2_{\tau_r}), r \geqslant 0,$$

defines a 2-dimensional Brownian motion in the filtration $(\mathcal{G}_r)_{r \geqslant 0}$ defined by $\mathcal{G}_r = \mathcal{F}_{\tau_r}$ for all $r \geqslant 0$. [You may use any result from the course provided that it is clearly stated.]

(b) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ be a filtered probability space such that the filtration $(\mathcal{F}_t)_{t \geqslant 0}$ is right-continuous and $\mathcal{F}_0$ contains all zero probability events. Let $(\beta_t, t \geqslant 0)$ and $(\theta_t, t \geqslant 0)$ be two independent $(\mathcal{F}_t)_{t \geqslant 0}$-Brownian motions in $\mathbb{R}$, and define a process $(Z_t, t \geqslant 0)$ with values in the complex plane as follows: let $R_t = \exp(\beta_t)$, and let

$$Z_t = R_t e^{i\theta_t}, \quad t \geqslant 0.$$

Let X_t and Y_t denote respectively the real and imaginary parts of Z_t. Show that X and Y are continuous local martingales and that

$$[X]_t = [Y]_t \quad \text{and} \quad [X, Y]_t = 0$$

for all $t \geqslant 0$. Show that $[X]_\infty = \infty$ almost surely. [*Hint:* the recurrence and the strong Markov property of $(\beta_t, t \geqslant 0)$, together with the law of large numbers, may be helpful for this result.]

(c) Deduce from (a) that there exists $(\tau_r, r \geqslant 0)$ such that $(Z_{\tau_r}, r \geqslant 0)$ is a two-dimensional Brownian motion started at $z_0 = (1, 0)$ in an appropriate filtration. Use this to show that Brownian motion in $\mathbb{R}^2$ started from z_0 almost surely never hits 0 but comes arbitrarily close to 0 on an unbounded set of times.

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(a) Let M be a continuous local martingale and let $Z = \mathcal{E}(M)$, where $\mathcal{E}(M)_t = \exp(M_t - (1/2)[M]_t)$ denotes the exponential local martingale associated with M. Show that $dZ_t = Z_t dM_t$. Suppose now that Z and Z' are two strictly positive processes such that $Z_0 = Z'_0$ and

$$dZ_t = Z_t dM_t; \quad dZ'_t = Z'_t dM_t.$$

By considering $\ln(Z'_t) - \ln(Z_t)$, show that Z and Z' are indistinguishable.

(b) Let $\mathbb{P}$ denote the Wiener measure (i.e., the law of a Brownian motion $(X_t, t \geqslant 0)$), and let $\mathbb{Q}$ be a measure which is absolutely continuous with respect to $\mathbb{P}$ on $\mathcal{F}_t$ for every $t > 0$, where $(\mathcal{F}_t)_{t \geqslant 0}$ is the filtration generated by X. We denote by Z_t the Radon-Nikodyn derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{F}_t$: that is,

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z_t.$$

Show that $(Z_t, t \geqslant 0)$ is a martingale in $(\mathcal{F}_t)_{t \geqslant 0}$. Assume that $Z_t = h(X_t, t)$, where $h(x, t)$ is a given positive $\mathcal{C}^2$ function. Deduce that $D_t h + D_{xx} h = 0$, where $D_t h$, $D_{tt} h$ (resp. $D_x h$, $D_{xx} h$) denote the first and second derivatives of h with respect to t (resp. x).

(c) Using the same notations as in (b), show that $dZ_t = Z_t dM_t$ where M is defined by:

$$M_0 = 0; \quad dM_t = \frac{D_x h(X_t, t)}{h(X_t, t)} dX_t.$$

Deduce from this and the result in (a) that $Z_t = \mathcal{E}(M)_t$. [You may use without proof the fact that $Z_0 = 1$ almost surely].

Conclude that the semimartingale decomposition of X under $\mathbb{Q}$ is

$$X_t = B_t + \int_0^t \frac{D_x h(X_s, s)}{h(X_s, s)} ds,$$

where B is a Brownian motion. [*Hint:* You may use Girsanov's theorem without proof, provided that you state it clearly and verify the assumptions carefully].

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(a) Let $(B_t, t \geqslant 0)$ be a one-dimensional Brownian motion and let $Y_t = e^{B_t + t/2}$. Show that Y solves a certain stochastic differential equation, whose coefficients should be determined. Does pathwise uniqueness hold for this equation? Using the Dubins-Schwartz theorem, show that there exists a Brownian motion $(\beta_t, t \geqslant 0)$ such that

$$Y_t = 1 + \beta_{H_t} + \int_0^t \frac{1}{Y_s} dH_s \qquad (1)$$

where $H_t = \int_0^t e^{2B_s + s} ds$.

(b) The notations are the same as in (a). Let $J_t = \inf\{s > 0 : H_s > t\}$. Deduce from the above that if $X_t = Y_{J_t}$, then

$$X_t = 1 + \beta_t + \int_0^t \frac{1}{X_s} ds. \qquad (2)$$

[*Hint:* note that $H_{J_t} = t$ for all $t \geqslant 0$ and use a change of variable in the integral appearing in the right-hand side of (1).]

(c) Let $(W_t, t \geqslant 0)$ be a Brownian motion in $\mathbb{R}^3$ started from $W_0 = (1, 0, 0)$, and let $|W_t|$ denote the Euclidean norm of W_t. Show that $|W|$ is also a solution of (2) and deduce that $\lim_{t\to\infty} |W_t| = \infty$ almost surely. [You may assume without proof that there is uniqueness in distribution for the solutions of (2).]

Show further that

$$\lim_{t\to\infty} \frac{\log(|W_t|)}{\log(t)} = \frac{1}{2},$$

almost surely. Assuming without proof the existence of this limit, explain briefly how you could have guessed its value using a different method.

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Throughout this question, we fix a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ satisfying the *usual conditions*: the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ is right-continuous and $\mathcal{F}_0$ contains all events of probability zero.

(a) Let $\sigma, b : \mathbb{R} \to \mathbb{R}$ be two measurable and locally bounded functions, and suppose that an adapted continuous stochastic process $(X_t, t \geqslant 0)$ is a solution to the stochastic differential equation:

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt \tag{1}$$

where $(B_t, t \geqslant 0)$ is a one-dimensional $(\mathcal{F}_t)_{t\geqslant 0}$-Brownian motion. Assume that $s : I \to \mathbb{R}$ is a $\mathcal{C}^2$ function on an interval I such that

$$\frac{1}{2}s''(x)\sigma^2(x) + s'(x)b(x) = 0, \quad x \in I. \tag{2}$$

Show that $Y_t = s(X_{t\wedge T})$ is a local martingale, where $T = \inf\{t \geqslant 0 : X_t \notin I\}$. Such a function s is called a scale function for (1) on I. Deduce that if $a < x < b$ with $a, b \in I$, and $X_0 = x$ almost surely, then

$$\mathbb{P}(T_b < T_a) = \frac{s(x) - s(a)}{s(b) - s(a)},$$

where for all $y \in \mathbb{R}$, $T_y = \inf\{t \geqslant 0 : X_t = y\}$.

(b) Let $a > 0$ with $a \neq 1/2$, and assume that X is a positive solution to the stochastic differential equation

$$dX_t = dB_t + \frac{a}{X_t}dt. \tag{3}$$

Show that for all $\epsilon > 0$, $s(x) = x^{-2a+1}$ is a scale function for (3) on $[\epsilon, \infty)$. Conclude that for all $x > 0$, if $X_0 = x$, then $T_0 = \infty$ almost surely if $a > 1/2$, while $T_0 < \infty$ almost surely if $a < 1/2$.

(c) Assume that $a > 1/2$. Show that for every $\epsilon > 0$ and for every driving Brownian motion B, there exists a unique process $(X^\epsilon_t, t \geqslant 0)$ which satisfies (3) for all $t < T^\epsilon_\epsilon$, where for all $\epsilon > 0$, $T^\epsilon_\epsilon = \inf\{t \geqslant 0 : X^\epsilon_t = \epsilon\}$. Show that T^ϵ_ϵ is nondecreasing as $\epsilon \to 0$, and let $T = \lim_{\epsilon\to 0} T^\epsilon_\epsilon$. Deduce that one can construct a process $(X_t, t \geqslant 0)$ which is a solution of (3) for all $t < T$. Show that necessarily $T = \infty$ almost surely. Conclude that in the case $a > 1/2$ there exists a strong solution to (3) for every driving Brownian motion B and that path-wise uniqueness holds.

END OF PAPER