M. PHIL. IN STATISTICAL SCIENCE

Friday 30 May 2008 1.30 to 4.30

STOCHASTIC NETWORKS

*Attempt **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A telephone banking facility has N incoming lines and a single (human) operator. Calls to the facility are initiated as a Poisson process of rate ν, but calls initiated when all N lines are in use are lost. A call finding a free line has then to wait for the operator to answer. The operator deals with waiting calls one at a time, and takes an exponentially distributed length of time with parameter λ to check the caller's identity, after which the call is passed to an automated handling system for the caller to transact banking business, and the operator is freed to deal with another caller. The automated handling system is able to serve up to N callers simultaneously, and the time it takes to serve a call is exponentially distributed with parameter μ. All these lengths of time are independent of each other and of the Poisson arrival process.

Model the facility as a closed migration process with state $n = (n_1, n_2, n_3)$, where n_1 denotes the number of free lines, n_2 denotes the number of calls waiting or in service at the operator, n_3 denotes the number of calls at the automated handling system and $n_1+n_2+n_3 = N$. Identify the transition rates for the process and show that in equilibrium the proportion of calls lost is

$$H(N)\left(\sum_{n=0}^{N} H(n)\right)^{-1}$$

where

$$H(n) = \left(\frac{\nu}{\lambda}\right)^n \sum_{i=0}^{n} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!}.$$

Develop an expression for the proportion of calls lost when the single operator is replaced by two operators able to deal with incoming calls.

2 Derive Erlang's formula, $E(\nu, C)$, for the proportion of calls lost at a resource of capacity C offered a load of ν. State clearly any assumptions you make in your derivation.

Define a loss network with fixed routing, and describe briefly how the equations

$$B_j = E\Big(\sum_r A_{jr}\nu_r \prod_{i\in r-\{j\}} (1-B_i), \quad C_j\Big), \quad j = 1, 2, \dots J \tag{1}$$

arise as a natural approximation for the link blocking probabilities in the network.

Establish the existence and uniqueness of a solution to the equations (1).

3 Write an essay on mathematical models of random access schemes. Your essay should cover the slotted infinite-population ALOHA scheme, and at least one scheme where the retransmission probability of a packet is not constant.

4 Derive Chernoff's bound

$$\mathbb{P}\{Y \geqslant 0\} \leqslant \inf_{s \geqslant 0} \mathbb{E}[e^{sY}]$$

for any real-valued random variable Y. Let $J, n_1, n_2, \ldots, n_J$ be non-negative integers. Let

$$Z = \sum_{j=1}^{J} \sum_{i=1}^{n_j} X_{ji}$$

where X_{ji} are independent random variables with

$$\alpha_j(s) = \frac{1}{s} \log \mathbb{E}[e^{sX_{ji}}] \quad i = 1, 2, \ldots, n_j \ .$$

Show that, for $C, \gamma > 0$,

$$\sum_{j=1}^{J} n_j \alpha_j(s) \leqslant C - \frac{\gamma}{s} \Rightarrow \mathbb{P}\{Z \geqslant C\} \leqslant e^{-\gamma},$$

and briefly discuss the interpretation of $\alpha_j(s)$ as an *effective bandwidth*.

Show that $\alpha_j(s)$ is increasing in s, and that it lies between the mean and the peak of X_{ji}, that is

$$\mathbb{E}X_{ji} \leqslant \alpha_j(s) \leqslant \sup\{x : \mathbb{P}\{X_{ji} > x\} > 0\}.$$

5 Let J be a set of resources, and R a set of routes, where a route $r \in R$ identifies a subject of J. Let C_j be the capacity of resource j, and suppose the number of flows in progress on each route is given by the vector $n = (n_r, r \in R)$. Define a proportionally fair rate allocation.

Consider a network with resources $J = \{1, 2\}$, each of unit capacity, and routes $R = \{\{1\}, \{2\}, \{1, 2\}\}$. Given $n = (n_r, r \in R)$, find the rate x_r of each flow on route r, for each $r \in R$, under a proportionally fair rate allocation.

Suppose now that flows describe the transfer of documents through a network, that new flows originate as independent Poisson processes of rates ν_r, $r \in R$, and that document sizes are independent and exponentially distributed with parameter μ_r for each route $r \in R$. Determine the transition intensities of the resulting Markov process $n = (n_r, r \in R)$. Show that the stationary distribution of the Markov process $n = (n_r, r \in R)$ takes the form

$$\pi(n) = B \binom{n_{\{1\}} + n_{\{2\}} + n_{\{1,2\}}}{n_{\{1,2\}}} \prod_{r \in R} \left(\frac{\nu_r}{\mu_r}\right)^{n_r},$$

where B is a normalizing constant, provided the parameters $(\nu_r, \mu_r, r \in R)$ satisfy certain conditions. Determine these conditions, and also the constant B.

$$[\mathit{Hint}: \quad \sum_{k=0}^{\infty} \binom{m+k-1}{k} (1-p)^m p^k = 1 \quad \text{for} \quad 0 < p < 1,\ m > 0.\ ]$$

END OF PAPER