UNIVERSITY OF CAMBRIDGE

M. PHIL. IN STATISTICAL SCIENCE

Friday 1 June 2007 9.00 to 11.00

INTRODUCTION TO PROBABILITY

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let N be a random variable taking values in $\{0, 1, 2, \ldots\}$.

- Prove that

$$\mathbb{E}(N) = \sum_{n=0}^{\infty} \mathbb{P}(N > n).$$

(b) Let $T_1, T_2, \ldots$ be independent Bernoulli random variables with

$$\mathbb{P}(T_n = 0) = \frac{1}{n} = 1 - \mathbb{P}(T_n = 1).$$

Let $N = \inf\{n \geq 1 : T_n = 1\}$.

- Prove that $\mathbb{E}(N) = e = 2.718\ldots$.

(c) Let U and V be independent $N(0,1)$ variables, and $W = U + V$.

- What is the distribution of W? (No proof is needed.)
- Find the conditional expectations

$$\mathbb{E}(W|U > 0, V > 0) \text{ and } \mathbb{E}(W^2|U > 0, V > 0).$$

(d) Let X and Y be random variables with joint density

$$f_{X,Y}(x, y) = \frac{2ye^{-xy}}{\pi(1 + y^2)} \text{ for } x \geqslant 0,\ y \geqslant 0$$

- Compute $\mathbb{E}(X|Y = y)$.

2 (a) Let $(N_t)_{t>0}$ be a continuous-time Markov process on the state space $\{0, 1, 2, \dots\}$ with generator $G = (g_{i,j})_{i,j\geq 0}$ given by $g_{i,i} = -\lambda$ and $g_{i,i+1} = \lambda$, for some constant $\lambda > 0$.

- Write down the forward Kolmogorov equations for the transition probabilities $p_{0,j}(t) = \mathbb{P}(N_t = j | N_0 = 0)$.
- Assuming the uniqueness of the solution to these forward equations, verify that conditional on the event $\{N_0 = 0\}$, the random variable N_t has a Poisson distribution with a parameter to be determined.

(b) Now assume $N_0 = 0$, and let $T_k = \inf\{t \geq 0 : N_t = k\}$ for each $k = 1, 2, \dots$.

- Prove that T_1 is an exponential random variable.
- Find the moment generating function of T_k.

(c) Let K be a geometric random variable such that $\mathbb{P}(K = k) = p(1-p)^{k-1}$ for $k \geqslant 1$, and assume K and $(N_t)_{t\geqslant 0}$ are independent. Define the random variable T_K by

$$T_K = \inf\{t \geqslant 0 : N_t = K\}.$$

- Find the moment generating function T_K.
- What is the distribution of T_K?

3 (a) Consider two boxes, labelled A and B. Initially, there are M marbles in box A and none in box B. Each minute afterwards, one of the M marbles is chosen uniformly at random and is moved to the opposite box. Let K_n denote the number of marbles in box A at time n, so that $K_0 = M$.

- Find the transition probabilities of the Markov chain $(K_n)_{n\geq 0}$.
- What is the invariant distribution?
- Compute $\mathbb{E}(T)$, where $T = \inf\{n \geq 1 : K_0 = M\}$ is the next time that all of the balls are in box A.

(b) Let $X_1, X_2, \dots$ be a sequence of independent random variables with

$$\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = \frac{1}{2}$$

for all n. Let $S_0 = 0$ and $S_n = X_1 + \dots + X_n$.

- Prove that the Markov chain $(S_n)_{n\geq 0}$ is a recurrent. You may use Stirling's formula:

$$\frac{n!}{\sqrt{2\pi} n^{n+1/2} e^{-n}} \to 1$$

as $n \to \infty$.

4 (a) Let $(X_n)_{n\geq 0}$ be a martingale relative to a filtration $(\mathcal{F}_n)_{n\geq 0}$. Let $(Y_n)_{n\geq 0}$ be a sequence of bounded random variables such that Y_n is $\mathcal{F}_n$-measurable for all $n \geq 0$. Let $M_0 = 0$ and

$$M_n = \sum_{k=0}^{n-1} Y_k(X_{k+1} - X_k)$$

for $n \geq 1$.

- Prove that $(M_n)_{n\geq 0}$ is a martingale.

(b) ◦ State the L_2 martingale convergence theorem.

(c) Let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables with $\mathbb{E}(Y_i) = 0$ and $\mathrm{Var}(Y_i) = 1$ for all $i \geq 1$, and let

$$S_n = \frac{1}{2}Y_1 + \frac{1}{4}Y_2 + \ldots + \frac{1}{2^n}Y_n$$

for $n \geq 1$.

- Find $\mathbb{E}(S_n)$ and $\mathrm{Var}(S_n)$ for each $n \geq 1$.
- Prove that there exists a random variable S such that $S_n \to S$ in L_2.

5 (a) Let $X_1, X_2, \ldots$ be a sequence of independent random variables each uniformly distributed uniformly $[0, 1]$. Let $M_n = \min\{X_1, \ldots, X_n\}$.

- Prove that nM_n converges in distribution to an exponential random variable with parameter one.

(b) Let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed random variables with $\mathbb{E}(Y_k) = \mu$ and $\mathrm{Var}(Y_k) = \sigma^2$ for all $k \geq 1$, where μ and σ are finite constants.

- State and prove the weak law of large numbers for $Y_1, Y_2, \ldots$.
- State (without proof) the central limit theorem for $Y_1, Y_2, \ldots$.

(c) Let $B_1, B_2, \ldots$ be sequence of events and let the event B be defined by

$$B = \bigcap_{N\geq 1} \bigcup_{n\geq N} B_n = \{B_n \text{ infinitely often}\}.$$

- Prove the first Borel-Cantelli lemma: if $\sum_{k=1}^{\infty} \mathbb{P}(B_k) < \infty$ then $\mathbb{P}(B) = 0$.

END OF PAPER