UNIVERSITY OF CAMBRIDGE

M. Phil. in STATISTICAL SCIENCE

Monday 12 June, 2006 1.30 to 3.30

SPREAD OF EPIDEMICS AND RUMOURS

Attempt ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The population consists of only susceptibles and infectives. A susceptible that is infected will stay infectious indefinitely.

We suppose that the population is closed, i.e. $X(t)+Y(t)=n$ for all $t \geqslant 0$ where $X(t)$ and $Y(t)$ denote the number of susceptibles and infectives at time t, with the initial condition $X(0)=an$ and $Y(0)=(1-a)n$, $a \in (0,1)$. At time t, a susceptible individual turns infected at rate $\lambda Y(t)/n$ where $\lambda > 0$.

(a) Let $X_n(t) = X(t)/n$ be the fraction of susceptible individuals at time t. Show that

$$X_n(t) \;=\; a - \int_0^t \lambda X_n(s)[1 - X_n(s)]\, ds - \epsilon_n(t)\,,$$

where $\epsilon_n(t)$ is expressed in terms of a standard Poisson process.

(b) Prove that

$$\lim_{n\to\infty} \sup_{0\leqslant t\leqslant T} |X_n(t) - x(t)| \;=\; 0\,, \quad \text{a.s.}$$

where

$$x(t) \;=\; \frac{ae^{-\lambda t}}{1 - a(1 - e^{-\lambda t})}\,.$$

[Hints: You may assume without proof the following results:

- Let $\mathcal{N}$ be a standard Poisson process and ϵ a positive constant. Then for $T > 0$,

$$\mathbb{P}\Big(\sup_{0\leqslant t\leqslant T} |\mathcal{N}(t) - t| \geqslant T\epsilon\Big) \leqslant 2e^{-Th(\epsilon)}\,,$$

where $h(t) = (1+t)\log(1+t) - t$.

- Gronwall's lemma: Let f be a real-valued function satisfying

$$f(t) \leqslant \alpha + \beta \int_0^t f(s)ds\,, t \geqslant 0\,.$$

Then $f(t) \leqslant \alpha e^{\beta t}$, *for* $t \geqslant 0$.]

2 Fix $p \in (0,1)$ and let $G(n,p)$ be the Erdös-Rényi random graph with n nodes, where each pair of nodes is connected independently with probability p (here p is constant and does not depend on n).

(a) Show that

$$\mathbb{P}\left(G(n,p) \text{ is disconnected}\right) \leqslant \sum_{s=1}^{[n/2]} \left(n(1-p)^{n/2}\right)^s ,$$

and deduce that

$$\lim_{n\to\infty} \mathbb{P}\left(G(n,p) \text{ is connected}\right) = 1 .$$

[*Hint: Find an upper bound on the probability that the graph admits an isolated component of size s, for $s \in \{1,\dots,[n/2]\}$.*]

(b) In the following, we are interested in the diameter of $G(n,p)$ for $p \in (0,1)$.

(i) For X a random variable in $\{0,1,2,\dots\}$ show that

$$\mathbb{P}(X > 0) \leqslant \mathbb{E}(X) .$$

(ii) Let X be the number of pairs of nodes with no common neighbour. Show that

$$\mathbb{E}(X) = \binom{n}{2}(1-p^2)^{n-2} .$$

(iii) Deduce that

$$\lim_{n\to\infty} \mathbb{P}\left(\text{Diameter of } G(n,p) \leqslant 2\right) = 1 .$$

3 We consider an Erdös-Renyi random graph $G(n,p)$ with n nodes, where each pair of nodes is connected independently with probability p. Given i,j,k three distinct nodes in $G(n,p)$, we say that the graph contains the triangle (i,j,k) if the links (i,j), (j,k) and (i,k) are present in $G(n,p)$. Let X be the number of triangles in $G(n,p)$.

(a) Prove that if

$$\lim_{n\to\infty} np = 0\,,$$

then the number of triangles in $G(n,p)$ is zero with probability tending to 1.

[*Hint: Use the following inequality:* $\mathbb{P}\left(X>0\right) \leqslant \mathbb{E}(X)$.]

(b) Assume that

$$\lim_{n\to\infty} np \;=\; \infty\,.$$

(i) Note that

$$X \;=\; \sum X_{ijk}\,,$$

where $X_{ijk}=1$ if the triangle ijk is present in $G(n,p)$ and $X_{ijk}=0$ otherwise, and where the sum is over distinct nodes i, j and k.

Show that

$$\mathbb{E}(X^2) \;=\; \sum_{l=0}^{3}\binom{n}{3}\binom{3}{l}\binom{n-3}{3-l}p^{2\binom{3}{2}-\binom{l}{2}}\,.$$

(ii) Using Chebyshev's inequality, establish the following inequality

$$\mathbb{P}\left(X=0\right) \leqslant \frac{\mathrm{Var}(X)}{(\mathbb{E}X)^2}\,,$$

where $\mathrm{Var}(X)$ is the variance of X.

(iii) Prove that

$$\lim_{n\to\infty} \mathbb{P}\left(\text{there is at least one triangle in } G(n,p)\right) \;=\; 1\,.$$

4 We consider an undirected graph $G = (V, E)$ where $V = \{1, \ldots, n\}$ is the set of nodes and $E \subset V \times V$ is the set of edges and let A be its adjacency matrix, i.e. $A_{ij} = A_{ji} = 1$ if i and j are connected in G (i.e. $(i, j) \in E$) and 0 otherwise. We assume that each node can be in one of three states: susceptible, infected or removed, and that initially there is at least one infected node.

The evolution of the epidemic is described by the following discrete-time model: an infected node stays infected for a unit of time during which it infects each of its neighbours with probability β and is then removed.

(a) Let $X_i(k)$ be the indicator that node i is infected at time k, and $Y_i(k)$ the indicator that node i is removed at time k. Show that

$$\mathbb{P}\left(Y_i(\infty) = 1\right) = \mathbb{P}\left(\text{node } i \text{ ever gets infected}\right) \leqslant \sum_{k=0}^{\infty} \sum_{j=1}^{n} \beta^k (A^k)_{ij} X_j(0) .$$

(b) Deduce that

$$\mathbb{E}\left[\sum_{i=1}^{n} Y_i(\infty)\right] \leqslant \sum_{k \geqslant 0} (1, \ldots, 1)(\beta A)^k X(0) ,$$

where

$$X(0) = \begin{pmatrix} X_1(0) \\ \vdots \\ X_n(0) \end{pmatrix} .$$

(c) Let ρ be the spectral radius of the adjacency matrix A (the eigenvalue of A with the largest absolute value) and suppose that $\beta\rho < 1$. Show that

$$\mathbb{E}\left[\sum_{i=1}^{n} Y_i(\infty)\right] \leqslant \frac{1}{1 - \beta\rho} \sqrt{n \sum_{i=1}^{n} X_i(0)} .$$

Deduce a sufficient condition for a small outbreak.

END OF PAPER