

M. PHIL. IN STATISTICAL SCIENCE

Friday 9 June, 2006 9 to 12

STOCHASTIC CALCULUS AND APPLICATIONS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

*Cover sheet
Treasury Tag
Script paper*

SPECIAL REQUIREMENTS

None

**You may not start to read the questions
printed on the subsequent pages until
instructed to do so by the Invigilator.**

1 (a) Define what it means for a process to be a local martingale.

(b) Suppose that X is a continuous local martingale of finite variation, with $X_0 = 0$. Show that $X \equiv 0$ almost surely. (You may assume that the total variation process of X is adapted and continuous.)

(c) Deduce that if Y is another continuous local martingale then there can be at most one continuous adapted increasing process A such that $Y^2 - A$ is again a continuous local martingale.

(d) Suppose now that H is a simple process, i.e.

$$H = \sum_{k=0}^{n-1} Z_k \mathbb{1}_{(t_k, t_{k+1}]},$$

where $n \in \mathbb{N}$, $0 = t_0 < t_1 < \dots < t_n < \infty$ and Z_k is a bounded \mathcal{F}_{t_k} -measurable random variable for each k . Suppose that M is an L^2 -bounded martingale with quadratic variation process $[M]$. Define the stochastic integral $H \cdot M$ and prove that

$$\mathbb{E}[(H \cdot M)_\infty^2] = \mathbb{E}[(H^2 \cdot [M])_\infty].$$

(You may assume that the Lebesgue-Stieltjes integral on the right-hand side is well-defined and that $H \cdot M$ and $M^2 - [M]$ are martingales.)

2 Suppose that B is a standard Brownian motion and that H is a locally bounded previsible process such that $\int_0^t H_s^2 ds$ is strictly increasing in t , $\int_0^t H_s^2 ds < \infty$ for all $t > 0$ and $\int_0^\infty H_s^2 ds = \infty$.

(a) Set $T = \inf\{t \geq 0 : \int_0^t H_s^2 ds > \sigma^2\}$, where $\sigma \neq 0$. Prove that

$$\int_0^T H_s dB_s \sim N(0, \sigma^2).$$

(b) State the Dubins-Schwarz theorem for a local martingale M .

(c) Using part (a), or otherwise, prove the Dubins-Schwarz theorem in the special case where $M_t = \int_0^t H_s dB_s$.

3 (a) Suppose that W is a Brownian motion and set

$$A_t = \int_0^t \operatorname{sgn}(W_s) dW_s,$$

where

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Show that A is another Brownian motion and that if $V_t = W_t^2$ then

$$dV_t = 2\sqrt{V_t}dA_t + dt$$

(here, \sqrt{x} is the non-negative square root of $x \in [0, \infty)$).

(b) Let $B^{(1)}$ and $B^{(2)}$ be independent Brownian motions and suppose that $\alpha \geq 0$ and $\beta \geq 0$ are constants. Let X satisfy

$$dX_t = 2\sqrt{X_t}dB_t^{(1)} + \alpha dt, \quad X_0 = x \geq 0$$

and let Y satisfy

$$dY_t = 2\sqrt{Y_t}dB_t^{(2)} + \beta dt, \quad Y_0 = y \geq 0.$$

Show that $Z = X + Y$ satisfies

$$dZ_t = 2\sqrt{Z_t}dB_t + \gamma dt, \tag{*}$$

where B is another Brownian motion and γ is a constant which you should determine.

(c) Suppose that Z is a solution to the stochastic differential equation (*) for some $\gamma \geq 2$, with $Z_0 = r^2$ and $r > 0$. Set $R_t = \sqrt{Z_t}$ and find a stochastic differential equation (which we will refer to as **(SDE)**) satisfied by R , at least until time $\zeta = \inf\{t \geq 0 : R_t = 0\}$.

(d) What does it mean for uniqueness in law to hold for a stochastic differential equation? Assume that uniqueness in law holds for **(SDE)** and suppose also that $\gamma \in \mathbb{Z}$ and $\gamma \geq 2$. Argue carefully that any solution to **(SDE)** must have the same distribution as the Euclidean norm of a γ -dimensional Brownian motion started from the sphere $\{x \in \mathbb{R}^\gamma : |x| = r\}$.

4 (a) Suppose that $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ and $b : \mathbb{R} \rightarrow \mathbb{R}$ are Lipschitz functions. Prove that there is pathwise uniqueness for the stochastic differential equation

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt.$$

(You may use Gronwall's lemma without proof.)

(b) Henceforth, consider the special case

$$dX_t = dB_t + X_t dt, \quad X_0 = 0.$$

By means of an exponential integrating factor, find the (pathwise unique) solution.

(c) Let $T = \inf\{t \geq 0 : X_t = 1 \text{ or } X_t = -1\}$. Suppose that under the probability measure $\tilde{\mathbb{P}}$, X is a Brownian motion. Using Girsanov's theorem, find a new probability measure \mathbb{P} , absolutely continuous with respect to $\tilde{\mathbb{P}}$, such that B is a Brownian motion under \mathbb{P} , at least until time T .

(d) Show that

$$\mathbb{P}(T \leq t) \geq \exp\left(\frac{1}{2} - t\right) \tilde{\mathbb{P}}(T \leq t).$$

(You may find it helpful to use Itô's formula to give an alternative expression for $\int_0^t X_s dX_s$.)

5 Consider, for $\varepsilon > 0$, the unique solution $u^\varepsilon \in C_b^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ of the Cauchy problem

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t} = L^\varepsilon u^\varepsilon & \text{on } (0, \infty) \times \mathbb{R}^d, \\ u^\varepsilon(0, \cdot) = f & \text{on } \mathbb{R}^d. \end{cases}$$

Here,

$$L^\varepsilon = \frac{\varepsilon^2}{2} \Delta + b(x) \cdot \nabla + c(x),$$

with b a Lipschitz vector field on \mathbb{R}^d , $c \in C_b(\mathbb{R}^d)$, and $f \in C_b^2(\mathbb{R}^d)$. Fix $x_0 \in \mathbb{R}^d$ and let $(x_t)_{t \geq 0}$ be the unique solution to the differential equation $\dot{x}_t = b(x_t)$ starting from x_0 . Show that, for all $t \geq 0$, as $\varepsilon \downarrow 0$,

$$u^\varepsilon(t, x_0) \rightarrow f(x_t) \exp\left\{\int_0^t c(x_s) ds\right\}.$$

You may use any result from the course without proof, provided that you state it clearly.

6 State what is meant by a Markov jump process X with state-space (E, \mathcal{E}) and generator Q , adapted to a given filtration $(\mathcal{F}_t)_{t \geq 0}$.

Write μ for the jump measure of X on $(0, \infty) \times E$, given by

$$\mu = \sum_{X_t \neq X_{t-}} \delta_{(t, X_t)}.$$

State a general result which allows one to identify martingales associated with X in terms of μ .

Consider now the case where X is a birth process with rates $\lambda(1), \lambda(2), \dots$. Thus X has state-space \mathbb{N} and, for all $t \geq 0$ and $i \in \mathbb{N}$, at time t , conditional on $X_t = i$, X jumps to $i + 1$ at rate $\lambda(i)$. Fix $\theta \in \mathbb{R}$ and set

$$M_t = \exp \left\{ \theta X_t - (e^\theta - 1) \int_0^t \lambda(X_s) ds \right\}.$$

Show that, for any value of θ , M is a local martingale up to the explosion time ζ of X . Show further that, if the rates are uniformly bounded, then M is in fact a martingale.

END OF PAPER