UNIVERSITY OF CAMBRIDGE

M. PHIL. IN STATISTICAL SCIENCE

Monday 5 June, 2006 9 to 12

ADVANCED PROBABILITY

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In this exercise, we consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n, n \geqslant 0), P)$, and all definitions are understood with respect to this filtered space.

a) Let $(X_n, n \geqslant 0)$ be a submartingale which is bounded in L^1.

(i) Prove that for every $n \geqslant 0$, the sequence $(E[X_p^+|\mathcal{F}_n], p \geqslant n)$ is increasing and converges to an a.s. limit M_n.

(ii) Show that $(M_n, n \geqslant 0)$ is a non-negative martingale which is bounded in L^1, and conclude that X_n can be written in the form $M_n - Y_n$, where $(Y_n, n \geqslant 0)$ is a non-negative supermartingale which is bounded in L^1.

b) Let $(X_n, n \geqslant 0)$ be a supermartingale which is bounded in L^1. Show that X_n can be written in the form $M_n + Y_n$, where $(M_n, n \geqslant 0)$ is a uniformly integrable martingale, and $(Y_n, n \geqslant 0)$ is a supermartingale with limit 0 when $n \to \infty$.

2 Let $\Omega = \{(\omega_1, \omega_2, \ldots) : \omega_i \in \mathbb{R}, i \geqslant 1\}$ be the set of real-valued sequences. For $\omega \in \Omega$ and $n \geqslant 1$ we let $X_n(\omega) = \omega_n$, and we let $S_0 = 0$, $S_n = X_1 + \ldots + X_n$. We define $\mathcal{F}_n = \sigma(X_k, 1 \leqslant k \leqslant n)$ and $\mathcal{F} = \mathcal{F}_\infty$.

Let μ be a probability measure on $\mathbb{R}$. We let P be the unique measure on $(\Omega, \mathcal{F})$ under which the sequence $X_1, X_2, \ldots$ is independent and identically distributed with common distribution μ. We let P_n be the restriction of P to $\mathcal{F}_n$.

For $\lambda \geqslant 0$ we let $\phi(\lambda) = E[e^{\lambda X_1}]$, where E is the expectation associated with P. We assume that $\phi(\lambda)$ is finite for every $\lambda \geqslant 0$.

a) Show that under P, for every $\lambda \geqslant 0$ the process $M^\lambda = (\exp(\lambda S_n)/\phi(\lambda)^n, n \geqslant 0)$ is an $(\mathcal{F}_n, n \geqslant 0)$-martingale.

b) Let P_n^λ be the probability measure on $(\Omega, \mathcal{F}_n)$ which is absolutely continuous with respect to P_n with density

$$\frac{\mathrm{d}P_n^\lambda}{\mathrm{d}P_n} = M_n^\lambda.$$

Show that under P_n^λ, the random variables $X_1, \ldots, X_n$ are independent and identically distributed. Identify their common distribution μ^λ, and show that it has mean $\phi'(\lambda)/\phi(\lambda)$.

c) In this part, we assume that μ is supported by $\mathbb{Z}_- \cup \{1\} = \{\ldots, -3, -2, -1, 0, 1\}$. For $k \geqslant 0$ let

$$\tau_k = \inf\{n \geqslant 0 : S_n \geqslant k\}.$$

We let P^λ be the unique probability distribution on $(\Omega, \mathcal{F})$ under which $(X_n, n \geqslant 1)$ is independent and identically distributed with common distribution μ^λ, and we let E^λ be the expectation associated with P^λ.

Show that

$$P(\tau_k \leqslant n) = E^\lambda[(M_n^\lambda)^{-1} \mathbb{1}_{\{\tau_k \leqslant n\}}] = E^\lambda[(M_{\tau_k}^\lambda)^{-1} \mathbb{1}_{\{\tau_k \leqslant n\}}] = e^{-\lambda k} E^\lambda[\phi(\lambda)^{\tau_k} \mathbb{1}_{\{\tau_k \leqslant n\}}].$$

Assuming that there exists $\lambda_0 > 0$ such that $\phi(\lambda_0) = 1, \phi'(\lambda_0) > 0$, compute $P(\tau_k < \infty)$, and deduce the law of $\sup_{n \geqslant 0} S_n$ under P.

3 Let $(M_t, t \geqslant 0)$ be a continuous-time martingale with respect to a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geqslant 0), P)$, such that $(M_t, t \geqslant 0)$ is a non-negative process with continuous paths, and which converges a.s. to 0 as $t \to \infty$. Let $M^* = \sup_{t \geqslant 0} M_t$. We use the notation $P(A|\mathcal{G}) = E[\mathbb{1}_A|\mathcal{G}]$, where A is an event and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$.

a) Show that for every $x > 0$,

$$P(M^* \geqslant x|\mathcal{F}_0) = 1 \wedge (M_0/x).$$

[Hint: Use the stopped martingale $(M_{t \wedge T_x}, t \geqslant 0)$, where $T_x = \inf\{t \geqslant 0 : M_t \geqslant x\}$.]

b) Deduce that M^* has the same distribution as M_0/U, where U is uniform on $[0, 1]$ and independent of M_0.

c) Let $(B_t, t \geqslant 0)$ be a Brownian motion started at $B_0 = a > 0$. Give the distribution of $\sup_{0 \leqslant t \leqslant T_0} B_t$, where $T_0 = \inf\{t \geqslant 0 : B_t = 0\}$.

4 State and prove the reflection principle for the standard 1-dimensional Brownian motion.

Let $(B_t, t \geqslant 0)$ be a standard 1-dimensional Brownian motion defined on some probability space $(\Omega, \mathcal{F}, P)$. Use the reflection principle to show that $S_t = \sup_{0 \leqslant s \leqslant t} B_s$ has the same law as $|B_t|$ for every $t \geqslant 0$.

Let $a < b < c < d$ be non-negative real numbers. Show that

$$P\left(\sup_{a \leqslant t \leqslant b} B_t = \sup_{c \leqslant t \leqslant d} B_t\right) = 0.$$

5 Let $(B_t, t \geqslant 0)$ be a standard 1-dimensional Brownian motion. For $x \in \mathbb{R}$, let $T_x = \inf\{t \geqslant 0 : B_t = x\}$. Fix $a, b > 0$, and let $T = T_a \wedge T_{-b}$.

By considering processes of the form $(\exp(\lambda B_t - \lambda^2 t/2), t \geqslant 0)$, or otherwise, prove that for every $\lambda \in \mathbb{R}$,

$$E(e^{-\lambda^2 T/2} \mathbb{1}_{\{T = T_a\}}) = \frac{\sinh(\lambda b)}{\sinh(\lambda(a + b))},$$

and that

$$E(e^{-\lambda^2 T/2}) = \frac{\cosh(\lambda(a - b)/2)}{\cosh(\lambda(a + b)/2)}.$$

6 a) Write carefully the definition of a Poisson random measure on a measurable space $(E, \mathcal{E})$, with σ-finite intensity $\mu(\mathrm{d}x)$.

b) Fix $d \geqslant 1$, and let $\lambda(\mathrm{d}x)$ be Lebesgue measure on $\mathbb{R}^d$. We let $B(0, r)$ be the open Euclidean ball in $\mathbb{R}^d$ with centre 0 and radius $r \geqslant 0$, and we let $v_d = \lambda(B(0, 1))$.

Let $M(\mathrm{d}x)$ be a Poisson random measure on $\mathbb{R}^d$ with intensity $\lambda(\mathrm{d}x)$. If f is a non-negative measurable function and ν is a non-negative measure, we let $\nu(f) = \int f \mathrm{d}\nu$.

(i) Let $R = \sup\{r \geqslant 0 : M(B(0, r)) = 0\}$. Show that the distribution of R has a density and compute it.

(ii) Let $N_r = M(B(0, r))$ for $r \geqslant 0$. Let $f : \mathbb{R}^d \to \mathbb{R}_+$ be a continuous function with compact support. Compute $E[N_r \exp(-M(f))]$.

(iii) Show that the two quantities

$$E[\exp(-M(f))|N_r \geqslant 1] \qquad \text{and} \qquad \frac{E[N_r \exp(-M(f))]}{P(N_r \geqslant 1)}$$

have the same limit as $r \downarrow 0$, and compute this limit.

END OF PAPER