UNIVERSITY OF CAMBRIDGE

M. PHIL. IN STATISTICAL SCIENCE

Thursday 1 June 2006 9-11

INTRODUCTION TO PROBABILITY

*Attempt **THREE** questions. There are **FIVE** questions in total.*

Marks for each question are indicated on the paper in square brackets.
Each question is worth a total of 20 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $X_1, X_2, \ldots$ and X be random variables.

(a) What does it mean to say

(i) $X_n \to X$ in distribution?

(ii) $X_n \to X$ almost surely?

(iii) $X_n \to X$ in L_2 (or mean-square)?

(iv) $X_n \to X$ in probability? [6]

(b) Prove that $X_n \to 0$ in distribution if and only if $X_n \to 0$ in probability. [4]

(c) Let $Y_1, Y_2, \ldots$ be independent with $\mathbb{E}(Y_i) = \mu$ and $\mathrm{Var}(Y_i) = \sigma^2$ for all $i = 1, 2, \ldots$. Prove that $\dfrac{Y_1 + \ldots + Y_n}{n} \to \mu$ in probability. [4]

(d) Let $X_n \to X$ in probability. Use the first Borel-Cantelli lemma to prove that there exists an increasing sequence $k_1, k_2, \ldots$ of natural numbers such that $X_{k_n} \to X$ almost surely. [6]

2 Let $X_1, X_2, \ldots$ be independent exponential random variables with rate parameter λ (so that $\mathbb{E}(X_i) = 1/\lambda$ for all $i = 1, 2, \ldots$)

Let N be a Poisson random variable with intensity parameter μ and let U be a random variable uniformly distributed on $[0, 1]$. Suppose that $N, U, X_1, X_2, \ldots$ are independent. Let $Y = X_1 + \ldots + X_N$ and $Z = (X_1 + X_2)U$.

(a) Prove that the moment generating function of Y is $M_Y(t) = \exp\left(\frac{\mu t}{\lambda - t}\right)$ for $t < \lambda$. [6]

(b) Prove that Z is an exponential random variable with rate λ. [6]

(c) Find the conditional density of X_1 given that $X_1 + X_2 = 1$. [8]

3 Let $(X_n)_{n\geqslant 0}$ be a stochastic process taking values in a countable space S.

(a) What does it mean to say $(X_n)_{n\geqslant 0}$ is a Markov Chain? [4]

Let $(X_n)_{n\geqslant 0}$ be a homogeneous Markov Chain.

(b) What does it mean to say a state $i \in S$

(i) is recurrent (or persistent)?

(ii) is positive recurrent? [4]

(c) Prove that a state $i \in S$ is transient if $\sum_{n=0}^{\infty} p_{ii}(n) < +\infty$ where $p_{jk}(n) = \mathbb{P}(X_n = k | X_0 = j)$ is the n-step transition probability. [5]

(d) The guests of a dinner party are seated at a circular table. There are N guests, labelled $1, \dots, N$ according to their position at the table with guest 1 sitting beside guests 2 and N and guest i is sitting beside guests $i-1$ and $i+1$ for $i = 2, \dots, N-1$. Unfortunately, there is only one bottle of wine. Once a guest pours herself a glass, she passes the bottle to either of her two neighbours with equal probability, independently of the previous position of the bottle. Guest 1 opens the bottle and pours the first glass for herself. On average, how many glasses are poured from the time the bottle is opened until the bottle is passed back to guest 1? On average, how many glasses are poured from the time the bottle is opened until the bottle is passed to guest i with $i = 2, \dots, N$? [7]

4 Let $(X_n)_{n\geqslant 0}$ be a stochastic process and $(\mathcal{F}_n)_{n\geqslant 0}$ be a filtration.

(a) What does it mean to say $(X_n)_{n\geq 0}$ is a martingale relative to $(\mathcal{F}_n)_{n\geqslant 0}$? [6]

(b) Let $Y_1, Y_2, \dots$ be independent with $\mathbb{E}(Y_i) = 0$ and $\mathrm{Var}(Y_i) = 1$ for all $i = 1, 2, \dots$. Let $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$ and $S_n = Y_1 + \dots + Y_n$. Prove that $(X_n)_{n\geqslant 1}$ is a martingale relative to $(\mathcal{F}_n)_{n\geqslant 1}$ where $X_n = S_n^2 - n$. [7]

(c) Let $(X_n)_{n\geqslant 0}$ be a bounded martingale relative to $(\mathcal{F}_n)_{n\geqslant 0}$ and let τ be an almost surely finite stopping time. Prove that $\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$. [7]

5 Let $G = (g_{ij})_{i,j \geqslant 0}$ be a matrix with entries

$$g_{ij} = \begin{cases} \lambda & \text{if } j = i+1 \\ -\lambda & \text{if } j = i = 0 \\ -(\mu+\lambda) & \text{if } j = i \geqslant 1 \\ \mu & \text{if } j = i-1 \end{cases}$$

for $i, j = 0, 1, 2, \ldots$ and constants $\lambda, \mu \geqslant 0$.

Let $(X_t)_{t \geqslant 0}$ be a continuous time Markov chain with generator G.

(a) Prove that if $\lambda < \mu$ there exists a unique invariant measure for the chain. Find it. [8]

(b) What are the forward Kolmogorov equations for the transition probabilities $p_{ij}(t) = \mathbb{P}(X_t = j | X_0 = i)$? [6]

(c) Prove that if $\mu = 0$ and $X_0 = 0$ then the random variable X_t has the Poisson distribution with parameter λt. [6]

END OF PAPER