UNIVERSITY OF CAMBRIDGE

M. PHIL. IN STATISTICAL SCIENCE

Monday 6 June 2005 9 to 11

PROBABILITY

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let D be a circular disc of radius 1. A point is chosen uniformly at random in D. (This means that, if A is a subset of D, then the probability that the point falls in A equals area(A)/area(D).)

Let X be the distance of the point from the centre of the disc.

(i) Show that the distribution function of X is given by

$$\mathbb{P}(X \leqslant x) = \begin{cases} 0, & x < 0, \\ x^2, & 0 \leqslant x \leqslant 1, \\ 1, & x > 1. \end{cases}$$

Now suppose that n points are chosen independently, each uniformly at random in D.

Let L_n be the distance from the centre to the nearest point.

(ii) Find the distribution function and probability density function of L_n.

(iii) Show that nL_n^2 converges in distribution as $n \to \infty$, and find its limit.

2 Let $X_1, X_2, \ldots, X_n$ be random variables.

(i) Define $\mathrm{Cov}\big(X_1, X_2\big)$, the covariance of X_1 and X_2.

(ii) Assume that $\mathbb{E}\,(X_1) = ... = \mathbb{E}\,(X_n) = 0$. Show that

$$\mathrm{Var}(X_1 + \ldots + X_n) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2 \sum_{1 \leqslant i < j \leqslant n} \mathrm{Cov}(X_i, X_j). \qquad (*)$$

[*You may assume the random variables under consideration have finite second moments so that all quantities above are finite.*]

(iii) Give a short argument why $(*)$ remains true without the zero-mean assumption in (ii).

Let $W_0, W_1, \ldots, W_n$ be independent random variables, each with the uniform distribution on $[0, 1]$.

For $i = 1, 2, \ldots, n$, say that *a descent occurs at* i if $W_{i-1} > W_i$. Let A_i be the event that a descent occurs at i.

Let S be the total number of descents in $1, \ldots, n$, that is, the size of the set

$$\Big\{ i \in \{1, \ldots, n\} : A_i \text{ occurs } \Big\}.$$

(iv) Find $\mathbb{P}(A_i)$ for each i, and $\mathbb{P}(A_i \cap A_j)$ for each $1 \leqslant i < j \leqslant n$.

(v) Find the mean and variance of S.

3 (i) Let $\lambda > 0$. What is the Poisson distribution with parameter λ? Calculate the mean in terms of λ.

Let $(X_t)_{t \geqslant 0}$ denote the Poisson process on $\{0, 1, 2, ...\}$ of intensity λ and starting at 0.

(ii) State the distribution of X_t for fixed t.

(iii) Define the characteristic function ϕ_X of a random variable X. Show that the characteristic function of X_t is given by

$$\phi_{X_t}(u) = \exp\left[\lambda t\left(e^{iu} - 1\right)\right].$$

(iv) Let $(Y_j)_{j \geqslant 1}$ be independent identically distributed random variables, independent of $(X_t)_{t \geqslant 0}$ and with common probability density function f. Set

$$Z_t = \sum_{j=1}^{X_t} Y_j.$$

Show that the characteristic function of Z_t has form

$$\phi_{Z_t}(u) = \exp\left[\int_{\mathbb{R}} g(z, t, \lambda, u)\, f(z)\, dz\right],$$

for a function $g = g(z, t, \lambda, u)$ which you should determine.
[*Hint: condition on* $X_t = n$.]

4 (i) State precisely the assumptions and conclusion of the L^2-martingale convergence theorem, then prove it. [*Properties of the conditional expectation may be used without further justification.*]

(ii) Consider the symmetric simple random-walk on $\mathbb{Z}$: This is the process $(M_n)_{n \geqslant 0}$ defined as

$$M_0 = 0$$
$$M_{n+1} = M_n + X_{n+1}$$

where $(X_n)_{n \geqslant 1}$ are independent identically distributed random-variables with

$$\mathbb{P}[X_1 = +1] = \mathbb{P}[X_1 = -1] = 1/2.$$

Show that (M_n) is a martingale with respect to the filtration $(\mathcal{F}_n)$ generated by (X_n).

(iii) Does the symmetric simple random walk on $\mathbb{Z}$ satisfy the assumptions for the L^2-martingale theorem? Justify your answer.

5 (i) Consider a Markov chain $(Y_n)_{n \geqslant 0}$ on a countable state space S. Define recurrence and transience of a state $x \in S$. What does it mean to say that recurrence and transience are class properties? When does it make sense to say that $(Y_n)_{n \geqslant 0}$ is recurrent or, respectively, transient?

(ii) Consider

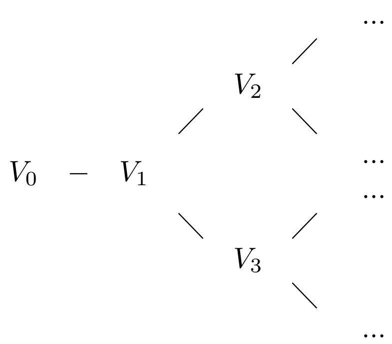
$$S_0 = 0, \;\; S_n = X_1 + ... + X_n \quad \text{for all} \;\; n \in \mathbb{N},$$

where $(X_i)_{i \geqslant 1}$ is a family of independent, identically distributed random variables with

$$\mathbb{P}\left[X_1 = 1\right] = p, \; \mathbb{P}\left[X_1 = -1\right] = 1 - p \quad \text{with} \;\; 1/2 < p < 1.$$

Using the strong law of large numbers or otherwise, show that S_n is transient.

(iii) The rooted binary tree is an infinite graph T with one distinguished vertex V_0 from which comes a single edge; at every other vertex there are three edges and there are no closed loops. The random walk on T jumps from a vertex along each available edge with equal probability. This is simply a random walk on the tree seen below. Show that this random walk is transient. [*Hint: consider the distance from* V_0.]



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