UNIVERSITY OF CAMBRIDGE

M. PHIL. IN STATISTICAL SCIENCE

Friday 4 June, 2004   9 to 12

QUANTUM INFORMATION THEORY

Attempt **FOUR** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define a classical linear (N, k) code C in terms of its generator and parity-check matrices. What is the dual code $C^{\perp}$? How are the generator and parity-check matrices related?

Explain how the concept of a classical dual code is used to define a quantum error-correcting code (CSS code). Discuss the error-correcting properties of a CSS code. Give an example of such a code.

2 Define a quantum operation (superoperator) on the set of density matrices. What are the properties to be satisfied so that a quantum operation would represent an allowed physical process? State the representation for a quantum operation referring to a coupling of a system and an environment. Draw a diagram illustrating this representation.

State the operator-sum (Kraus) representation for a quantum operation. Prove the equivalence of the two representations.

Prove that any quantum operation written in the Kraus form is linear, completely positive and trace-preserving.

3 a) State the generalised measurement postulate. When is a generalised measurement reduced to a projective measurement? Define a POVM and show how it is related to a generalised measurement. What is a pure POVM? State and prove the duality between POVMs and maximally mixed states.

b) Give an example of a positive map which does not satisfy the complete positivity property. Justify your answer.

4 Define the von Neumann entropy $S(\rho)$ of a density matrix ρ acting in a finite-dimensional Hilbert space $\mathcal{K}$. State and prove the bounds indicating the range of values of $S(\rho)$. Comment on the cases of equality in these bounds.

Prove that $S(\rho)$ is a concave function of ρ, i.e.

$$S(p_1\rho_1 + \ldots + p_m\rho_m) \geqslant p_1 S(\rho_1) + \ldots + p_m S(\rho_m)$$

for any collection of density matrices $\rho_1, \ldots, \rho_m$ and probabilities $p_1, \ldots, p_m$. Prove also that

$$S(p_1\rho_1 + \ldots + p_m\rho_m) \leqslant p_1 S(\rho_1) + \ldots + p_m S(\rho_m) - \sum_{i=1}^{m} p_i \log_2 p_i \,.$$

5 State and prove Schumacher's noiseless coding theorem.

6 Define a memoryless quantum depolarising channel. Show how this channel acts in the Bloch sphere representation where a single-qubit density matrix is associated with a point of a unit ball in $\mathbb{R}^3$.

State, without proof, the formulae for the un-assisted product-state capacity and the entanglement-assisted capacity of the depolarising channel. Draw the graphs showing how these capacities depend on the probability of error. Calculate the limiting ratio of these capacities when they both tend to zero.