UNIVERSITY OF CAMBRIDGE

M. PHIL. IN STATISTICAL SCIENCE

Thursday 27 May, 2004 1.30 to 4.30

ADVANCED PROBABILITY

Attempt **FOUR** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 a) State and prove Doob's maximal inequality.

b) Let $\xi_1, \xi_2, \ldots$ be independent random variables with $\mathsf{E}\xi_j = 0$ and $\mathsf{E}\left[{\xi_j}^2\right] < \infty$. Denote $S_n = \xi_1 + \ldots + \xi_n$. Prove Kolmogorov's inequality: for any $x > 0$,

$$\mathsf{P}\Big(\max_{0 \le m \le n} |S_m| \ge x\Big) \le \frac{\mathsf{E}\left[{S_n}^2\right]}{x^2}.$$

c) Let X_n be a martingale with $X_0 = 0$ and $\mathsf{E}\left[{X_n}^2\right] < \infty$. Show that for all $x > 0$

$$\mathsf{P}\Big(\max_{0 \le m \le n} X_m \ge x\Big) \le \frac{\mathsf{E}\left[{X_n}^2\right]}{\mathsf{E}\left[{X_n}^2\right] + x^2}.$$

[*Hint: Consider the process* $(X_n + c)^2$ *and optimize w.r.t.* c.]

2 a) Let X be a non-negative random variable with $\mathsf{E}\, X^2 < \infty$. Show that its Laplace transform, $\phi_X(\lambda) \equiv \mathsf{E}\, e^{-\lambda X}$, $\lambda \ge 0$, is differentiable for any $\lambda > 0$ and has the following decomposition:

$$\phi_X(\lambda) \equiv \mathsf{E}\, e^{-\lambda X} = 1 - \mathsf{E}\, X\, \lambda + \frac{\mathsf{E}\, X^2}{2}\lambda^2 + o(\lambda^2)$$

as $\lambda \to 0$.

b) Let B_t be a Brownian motion starting from 0. For $a > 0$, denote by T the exit time from the interval $(-a, a)$, ie, $T = \inf\Big\{t : B_t \notin (-a, a)\Big\}$. Show that

$$\phi_T(\lambda) = \frac{1}{\cosh\big(a\sqrt{2\lambda}\big)}$$

and deduce that $\mathsf{E}T = a^2$, $\mathsf{Var}T = 2a^4/3$.

3 a) Let ξ_k, $k \geq 1$, be integrable non-negative random variables satisfying

$$\mathsf{E}\Big(\xi_{n+1}|\xi_1, \ldots, \xi_n\Big) \leq \xi_n + \delta_n ,$$

where $\delta_n \geq 0$ are constants and $\sum \delta_n < \infty$. Show that ξ_n converges (a.s.) to a finite limit ξ.

b) Let $Y_k > 0$, $k \geq 1$, be a sequence of finite random variables adapted to the filtration $\big(\mathcal{F}_m\big)_{m \geq 0}$. Denote $U_1 = 0$, $U_k = \sum_{j=1}^{k-1} Y_j$ and, for $a > 0$, define

$$\tau_a = \min\Big\{ n : \sum_{j=1}^{n} Y_j > a \Big\}.$$

Show that $a - U_{n \wedge \tau_a}$ is a non-negative $\big(\mathcal{F}_m\big)_{m \geq 0}$-supermartingale for any $a > 0$.

c) Let $X_k > 0$, $Y_k > 0$, $k \geq 1$, be sequences of finite random variables defined on the same probability space and adapted to the filtration $\big(\mathcal{F}_m\big)_{m \geq 0}$. Suppose that X_n are integrable and, almost surely, we have $\sum_k Y_k < \infty$ and

$$\mathsf{E}\big(X_{n+1}|\mathcal{F}_n\big) \leq X_n + Y_n , \qquad n \geq 0 .$$

Show that X_n converges (a.s.) to a finite limit.

4 State and prove Kolmogorov's criterion of continuity of trajectories for continuous-time random processes.

Deduce that trajectories of Brownian motion are (a.s.) Hölder continuous of exponent α, for any $\alpha < 1/2$.

5 a) Assume n "stars" are located in the interval $[-n, n]$ on the real line. Their locations are independent, each being uniformly distributed in the interval. Each star has mass $m > 0$, and the gravitational constant is unity. The force which will be exerted on a unit mass at the origin (the field strength) is then

$$F_n = \sum_{j=1}^{n} \frac{m\,\mathsf{sign}(X_j)}{{X_j}^2},$$

where X_j is the coordinate of the j'th star and $\mathsf{sign}(x)$ is the usual sign function. Show that the distributions of F_n converge weakly as $n \to \infty$.

b) Suppose that the inverse-square attraction in a) were replaced by an inverse p'th power attraction. Show that for p satisfying $0 < 1/p < 2$, one gets convergence to the stable law of index $1/p$, ie.,

$$\mathsf{E}\exp\{itF_n\} \to \exp\{-c_p|t|^{1/p}\}, \qquad \text{as } n \to \infty,$$

with some constant $c_p > 0$ to be found.

c) Suppose that the attraction is as in a) but the number of stars in $[-n, n]$ is random, namely,

$$F_n = \sum_{j=1}^{M} \frac{m\,\mathsf{sign}(X_j)}{{X_j}^2},$$

where M is a Poisson random variable with parameter n, idependent of all X_j. Find the corresponding weak limit of the sequence F_n.

6 a) Let sequences X_n and Y_n of random variables converge in probability to random variables X and Y respectively. Show that $X_n + Y_n$ converges in probability to $X + Y$.

b) We say that a sequence of random variables converges weakly if their distributions converge weakly. Show by counterexample that the analogue of the statement in a) for weak convergence may not be true.

c) Let a sequence X_n converge weakly to a random variable X and let a sequence Y_n converge weakly to a constant random variable Y. Show that $X_n + Y_n$ converges weakly to $X + Y$.