UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 3 June, 2003 1:30 to 4:30

PAPER 32

Advanced Financial Models

Attempt **FOUR** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\boldsymbol{A}_0$ be a fixed vector in $\mathbb{R}^r$ and $\boldsymbol{A}_1$ a random vector taking values in $\mathbb{R}^r$. Prove that exactly one of the following alternatives (a) or (b) holds:

(a) there exists a vector $\boldsymbol{x} \in \mathbb{R}^r$ satisfying either

$$\begin{array}{lll} & (i) & \boldsymbol{x}^\top \boldsymbol{A}_0 \leqslant 0, \quad \boldsymbol{x}^\top \boldsymbol{A}_1 \geqslant 0 \quad \text{and} \quad \mathbb{P}\left(\boldsymbol{x}^\top \boldsymbol{A}_1 > 0\right) > 0 \\ \text{or} & (ii) & \boldsymbol{x}^\top \boldsymbol{A}_0 < 0 \quad \text{and} \quad \boldsymbol{x}^\top \boldsymbol{A}_1 \geqslant 0; \end{array}$$

(b) there exists a positive random variable ν, $\mathbb{P}(\nu > 0) = 1$, with $\mathbb{E}\,\|\nu \boldsymbol{A}_1\| < \infty$ such that $\boldsymbol{A}_0 = \mathbb{E}(\nu \boldsymbol{A}_1)$.

Explain briefly the relevance of this result to the characterization of lack of arbitrage in a one-period financial model.

2 In a multi-period, discrete-time stochastic model operating over times $r = 0, 1, \ldots, n$, let $\{\boldsymbol{S}_r\}$ denote the prices of the assets which are adapted to a filtration $\{\mathcal{F}_r\}$; let $\boldsymbol{X} = \{\boldsymbol{X}_r\}$ be a trading strategy with associated dividend sequence $\{D_r^{\boldsymbol{X}}\}$ and let $\{B_r\}$ be an appropriate discounting sequence. Develop the theory of the model to the point of establishing the relationship

$$B_r \boldsymbol{X}_r^\top \boldsymbol{S}_r = \mathbb{E}_{\mathbb{Q}} \left(\sum_{j=r+1}^{n} B_j D_j^{\boldsymbol{X}} \;\middle|\; \mathcal{F}_r \right),$$

for $0 \leqslant r \leqslant n-1$, where $\mathbb{Q}$ is an equivalent martingale probability.

Explain how this relationship leads to a pricing formula for any attainable contingent claim.

3 Let $T_{a,b}$ denote the first hitting time of the line $a + bs$ by a standard Brownian motion, where $a > 0$ and $-\infty < b < \infty$ and let $T_a = T_{a,0}$ represent the first hitting time of the level a.

For $\theta > 0$, by using the fact that $\mathbb{E}\left(e^{-\theta T_a}\right) = e^{-a\sqrt{2\theta}}$, or otherwise, derive an expression for $\mathbb{E}\left(e^{-\theta T_{a,b}}\right)$ for each b, $-\infty < b < \infty$.

Hence, or otherwise, show that, for $t > 0$,

$$\mathbb{P}\left(T_{a,b} \leqslant t\right) = e^{-2ab}\Phi\left(\frac{bt-a}{\sqrt{t}}\right) + 1 - \Phi\left(\frac{a+bt}{\sqrt{t}}\right).$$

Use this result, in the context of the Black–Scholes model, to derive the price at time 0 of a barrier digital call which pays 1 at time t_0 if and only if the stock price reaches a predetermined barrier $c > S_0$, at or before that time, where S_0 is the initial price of the stock.

4 Suppose that in the Black–Scholes model, the stock price at time t is S_t, the fixed interest rate is ρ and the volatility is σ. Show that $f(S_t, t)$ is the value of a self-financing portfolio if and only if the function $f = f(x,t)$ satisfies the Black–Scholes equation

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + \rho x \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} - \rho f = 0.$$

Now consider the case where the underlying asset pays a dividend continuously in time and which is proportional to the stock price, so that at the instant t the rate of dividend payment is θS_t per unit time, where θ is a constant. Determine how the Black–Scholes equation should be modified in this situation.

Suppose further that $q(x,t)$ denotes the price at time t of a claim paying $f\left(S_{t_0}\right)$ at time $t_0 > t$, when $S_t = x$. Show that $q(x,t) = e^{-\theta(t_0 - t)} p(x,t)$, where $p(x,t)$ is the price of the claim when no dividend is paid but the interest rate is $\rho - \theta$.

5 Write an essay on utility maximization in the context of the Black–Scholes model. Your account should include discussion of the case where the investor derives utility continuously over time in addition to the utility of his final wealth and it should also provide a detailed treatment of at least one example.

6 Write an essay on Gaussian models for interest rates. The essay should cover one, or more, of the following topics:

(i) one-factor models;

(ii) general random-field models;

(iii) pricing caps;

(iv) Markovian properties and stationarity of random-field models.

A careful treatment of any one of these topics may earn full marks.