UNIVERSITY OF CAMBRIDGE

M. P*HIL. IN* STATISTICAL SCIENCE

Tuesday 4 June 2002 1.30 to 3.30

INTERACTING PARTICLE SYSTEMS

*Attempt **THREE** questions*

*There are **four** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Describe the bond percolation model with parameter p on the square lattice $\mathbb{Z}^2$. What does it mean to say that an event associated with this process is *increasing*?

State the (Harris)–FKG inequality and the disjoint-occurrence inequality for two increasing events.

Let $\Lambda_n = [-n, n]^2$ and $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$, and let $g_k = P_p(0 \leftrightarrow \partial\Lambda_k)$ be the probability of an open path in the process joining the origin to some vertex of $\partial\Lambda_k$. Show that

$$g_n \leqslant g_{n-m} \sum_{x \in \partial\Lambda_m} P_p(0 \leftrightarrow x), \quad m \leqslant n.$$

Let $\chi(p)$ be the mean size of the open cluster containing the origin, and assume that p is such that $\chi(p) < \infty$. Show that there exists $\gamma > 0$ such that $g_k \leqslant e^{-\gamma k}$ for all k.

2 Let T_d be a homogeneous infinite tree in which each vertex has degree $d + 1$. Let ξ_t^A be the set of infected vertices at time t in the contact model on T_d with infection rate λ and death rate 1, under the assumption that the infected set at time 0 is the non-empty finite set A. The corresponding probability measure is written P_λ, with expectation E_λ.

Let $0 < \rho < 1$ and define $\nu_\rho(B) = \rho^{|B|}$, for a set B of vertices. Show that

$$\frac{d}{dt} E_\lambda\left(\nu_\rho\left(\xi_t^A\right)\right)\Big|_{t=0} \leqslant (1-\rho)\nu_\rho(A)\left[\frac{|A|}{\rho}(1 - \lambda\rho(d-1)) - 2\lambda\right].$$

Deduce that $E_\lambda(\nu_\rho(\xi_t^A))$ is non-increasing in t, if $\rho\lambda(d-1) \geqslant 1$.

Let $\lambda_1 = \inf\{\lambda : P_\lambda(\xi_t^{\{x\}} \neq \phi \text{ for all } t) > 0\}$, where x denotes a vertex of the tree. Show that $\lambda_1 < 1/(d-1)$.

To each vertex x of T_d is allocated an integer $g(x)$ in such a way that: if x, y are neighbours, then $g(y) = g(x) \pm 1$, and furthermore each x has exactly one neighbour y with $g(y) = g(x) - 1$. By considering the function $w_\rho(A) = \sum_{x \in A} \rho^{g(x)}$ or otherwise, where $0 < \rho < 1$ and $A \subseteq V$, show that

$$\lambda_2 = \inf\{\lambda : P_\lambda(x \in \xi_t^{\{x\}} \text{ for unbounded } t) > 0\}$$

satisfies $\lambda_2 \geqslant 1/\{2\sqrt{d}\}$.

3 Let $G = (V, E)$ be a finite graph and let $0 < p < 1$ and $q \in \{2, 3, \ldots\}$. On the product sample space $\{1, 2, \ldots, q\}^V \times \{0, 1\}^E$ we define the probability mass function

$$\mu(\sigma, \omega) = \frac{1}{Z} \prod_{e \in E} \{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)\},$$

for $\sigma \in \{1, 2, \ldots, q\}^V$, $\omega \in \{0, 1\}^E$, where $\delta_{r,s}$ is the Kronecker delta, and $\delta_e(\sigma) = \delta_{\sigma(x),\sigma(y)}$ where e is the edge with endvertices x and y. Here Z is a constant depending on p and q.

Show that the marginal mass functions $\mu_1(\sigma) = \sum_\omega \mu(\sigma, \omega)$, $\mu_2(\omega) = \sum_\sigma \mu(\sigma, \omega)$ are given by the Potts and random-cluster measures

$$\mu_1(\sigma) = \frac{1}{Z'} \exp\left(\beta \sum_e \delta_e(\sigma)\right), \text{ where } e^{-\beta} = 1 - p,$$

$$\mu_2(\omega) = \frac{1}{Z''} \left\{\prod_e p^{\omega(e)}(1-p)^{1-\omega(e)}\right\} q^{k(\omega)},$$

for constants Z', Z'', where $k(\omega)$ is the number of open clusters under ω.

Find the conditional mass function $\mu(\sigma \,|\, \omega)$ of σ given the edge-configuration ω. Deduce that, for $x, y \in V$,

$$\mu_1(\sigma(x) = \sigma(y)) - \frac{1}{q} = (1 - q^{-1})\ \mu_2(x \leftrightarrow y),$$

where $\{x \leftrightarrow y\}$ is the event that ω contains an open path from x to y.

4 Let $G = (V, E)$ be a finite regular graph (i.e., each vertex has the same number of neighbours). Particles inhabit the vertices in V, and each vertex may be occupied by no more than one particle at any time. Each particle jumps at rate 1, and when it jumps it does so to a neighbour chosen uniformly at random. If this neighbour is already occupied by a particle then the jump does not take place. You may assume the maximum amount of independence between jumps.

Let η_t^A be the set of vertices occupied by the particles at time t, where A is the set of their initial positions. Show that

$$P\big(\eta_t^A \supseteq B\big) = P\big(\eta_t^B \subseteq A\big)$$

for $A, B \subseteq V$.

For $0 \leqslant \rho \leqslant 1$, let μ_ρ be product measure on the configuration space $\{0, 1\}^V$ with density ρ; i.e., each vertex is occupied with probability ρ, independently of all other vertices. Show that, for all $0 \leqslant \rho \leqslant 1$, the measure μ_ρ is invariant for the above exclusion process.