UNIVERSITY OF CAMBRIDGE

M. PHIL. IN STATISTICAL SCIENCE

Thursday 31 May 2001   9 to 11

ACTUARIAL STATISTICS

Attempt any ***THREE*** *questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the *cumulant generating function* and the *cumulants* of a random variable Y. Show that the first and second cumulants are the mean and variance respectively. Let $T = \sum_{i=1}^{N} V_i$ where N has a Poisson distribution with mean ν, and $V_1, V_2, \ldots$ are independent and identically distributed random variables, independent of N. Show that the cumulant generating function of T is $\kappa_T(t) = \nu(M_V(t)-1)$ where $M_V(t)$ is the moment generating function of V_i. Hence or otherwise find $\mathbb{E}T$ and var$T$ in terms of ν and moments of V_i.

In a particular region, the number M of floods in a year has a Poisson distribution with mean ν. The total amount of claims arising from the i^{th} flood is $S_i = \sum_{j=1}^{N_i} X_{ij}, i = 1, \ldots, M$, where $N_1, N_2, \ldots$ are independent Poisson random variables with mean λ, X_{ij} denotes the j^{th} claim from flood i, the X_{ij}'s are independent with $\mathbb{E}X_{ij} = \mu, \text{var}X_{ij} = \sigma^2$ for all i, j. Assume $M, \{N_i\}, \{X_{ij}\}$ are independent, and let S be the total amount of claims arising from all the floods occuring in one year in this region. Find $\mathbb{E}S$ and var$S$.

Find the probability generating function of the total number N of claims in one year and determine whether or not N has a Poisson distribution.

Suppose that, independently for each claim, there is probability p that it is found not to be valid. Find the distribution of the number of valid claims arising from the i^{th} flood. Find the expectation of the total amount paid in a year on valid claims.

2 In a classical risk model, claims arrive in a Poisson process rate $\lambda > 0$, the claim sizes have density $f(x)$ and the premium income rate is $c > 0$. Let $\phi(u)$ be the probability of never being ruined when the initial capital is $u > 0$. Show that

$$\phi'(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_0^u \phi(u-x)f(x)dx.$$

Suppose that $f(x) = 3e^{-4x} + \frac{1}{2}e^{-2x}, x > 0$, and that the relative safety loading is $\rho = 3/5$. Show that

$$\phi'''(u) + 4\phi''(u) + 3\phi'(u) = 0.$$

Given $\phi(0) = \frac{\rho}{1+\rho}$, find $\phi(u)$.

3 Explain what is meant by a *credibility factor*.

Let $X_1, X_2, \ldots, X_n$ be the claim amounts on a particular risk in successive years and suppose these claim amounts depend on a risk parameter θ. Given θ, the X_i's are conditionally independent with $\mu(\theta) = \mathbb{E}(X_i|\theta)$ and $\sigma^2(\theta) = \text{var}(X_i|\theta)$ not depending on i. Let $m = \mathbb{E}(\mu(\theta)), a = \text{var}(\mu(\theta))$ and suppose $a > 0$. Find $\mathbb{E}X_i, \text{var}X_i$ and $\text{cov}(X_i, X_j)$ when $i \neq j$. Find also $\text{cov}[\mu(\theta), X_i]$.

Determine the credibility premium and write down the credibility factor. Discuss the behaviour of the credibility factor as $\mathbb{E}[\sigma^2(\theta)]$ increases.

Suppose data $\{X_{js}\}$ $j = 1, \ldots, n,\ s = 1, \ldots, k$ are available, where X_{js} is the claim amount in time period j on risk s. How may these data be used to obtain a numerical estimate of the credibility premium?

4 Explain how a No Claims Discount system can be modelled by a discrete-time Markov chain.

The cost of repair of a car after an accident is £L where L has density

$$f_L(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right\}, \quad x > 0.$$

Show that $\mathbb{P}(L > a) = 1 - \Phi(\frac{\log a - \mu}{\sigma})$ where Φ is the standard normal distribution function.

In a No Claims Discount system there are three discount levels 0%, 20% and 30%, and the full annual premium is \$$C$. If no claims are made in a year then the policyholder moves to the next higher level of discount (or remains at 30%). If one or more claim is made during a year then the policyholder moves down to (or remains at) the 0% level. Let p be the probability of no claims in a year. Suppose the cost of repair is L with density above. The policyholder decides whether or not to make a claim by considering the costs and premiums for the current year and the following two years, assuming no further accidents.

Find the transition matrix for the Markov chain for this system. Find the expected size of a claim, given that a claim is made, for a policyholder paying the full premium.