UNIVERSITY OF CAMBRIDGE

M. PHIL. IN STATISTICAL SCIENCE

Tuesday 5 June 2001 1.30 to 4.30

STOCHASTIC CALCULUS AND APPLICATIONS

Answer ***FOUR*** *questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $(\Omega, \mathcal{F})$ be a measurable space equipped with a filtration $(\mathcal{F}_t)_{t \geqslant 0}$. Explain what is meant by the previsible σ-algebra $\mathcal{P}$.

Consider the case where $\Omega = (0, \infty)$ and $\mathcal{F}$ is the Borel σ-algebra on Ω. Suppose that $(\mathcal{F}_t)_{t \geqslant 0}$ is the natural filtration of the process

$$X_t(\omega) = 1_{t \leqslant \omega}.$$

Show that

(a) $\mathcal{F}_t = \{B \in \mathcal{F} \ : \ B \subseteq (0, t) \quad \text{or} \quad B^c \subseteq (0, t)\}$,

(b) $\mathcal{P} = \{((\Omega \times T) \cap \{(\omega, t) : \omega \geqslant t\}) \cup (A \cap \{(\omega, t) : \omega < t\}) : A \in \mathcal{F} \otimes \mathcal{F}, T \in \mathcal{F}\}$.

2 Let M be a continuous L^2-bounded martingale and let H be a strictly simple process, that is, for some $0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n$ and bounded $\mathcal{F}_{t_k}$-measurable random variables Z_{t_k},

$$H_t = \sum_{k=0}^{n-1} Z_{t_k} 1_{t_k < t \leqslant t_{k+1}}.$$

Define the integral $(H \cdot M)_t$ and show that

$$\mathbb{E}[(H \cdot M)_\infty^2] \ = \ \mathbb{E} \int_0^\infty H_s^2 d[M]_s$$

where $[M]$ is the quadratic variation process of M.

Outline briefly how this identity allows extension of the notion of the integral $H \cdot M$ to all locally bounded previsible processes.

3 Let M be a continuous local martingale starting from 0.

(a) Show that, if f is a C^2 function and $f(M)$ is of finite variation, then $f(M) \equiv f(0)$.

(b) Show that, if A is a deterministic continuous finite variation process and $M^2 - A$ is a local martingale, then $M_t \sim N(0, A_t)$ for all $t \geqslant 0$.

4 Consider the stochastic differential equation in $\mathbb{R}$

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \qquad X_0 = 0.$$

Here $\sigma, b : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions and B is a Brownian motion. Suppose that, for some finite constant A, we have

$$|\sigma(x)|^2 + |b(x)|^2 \leqslant A(1 + |x|^2). \tag{$*$}$$

Show that, for any solution X,

$$\mathbb{E}\Big(\sup_{t \leqslant 1} |X_t|^2\Big) \leqslant e^{8A}.$$

Deduce that, even if σ and b are only locally Lipschitz, then condition (*) guarantees the existence of a solution $(X_t)_{t \leqslant 1}$ to the stochastic differential equation.

5 Consider the differential operator on $\mathbb{R}$

$$Lf(x) = \frac{x^2}{2} f''(x) + \frac{x}{2} f'(x) - \frac{1}{2} f(x).$$

Let $u \in C_b^{1,2}([0,\infty) \times \mathbb{R})$ solve the Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= Lu \quad \text{on} \quad [0,\infty) \times \mathbb{R}, \\ u(0,\cdot) &= g(\cdot) \quad \text{on} \quad \mathbb{R}. \end{aligned}$$

Show that, for the solution $(X_t)_{t \geqslant 0}$ of a certain stochastic differential equation, we have

$$u(t,x) = \mathbb{E}_x\big(e^{-t/2} g(X_t)\big), \qquad t \geqslant 0.$$

Hence obtain the fundamental solution $p(t,x,y)$ of the Cauchy problem for L, showing in particular that

$$p(t,1,y) = \frac{1}{\sqrt{2\pi t}} e^{-(\log y + t)^2/2t}, \quad y > 0.$$

6 In the playground of a large school there is a craze for paper-scissors-stone. This is a game for two players, each of whom simultaneously declares himself or herself to be one of paper, scissors or stone. The winner is decided as follows: scissors cut paper, stone blunts scissors and paper wraps up stone. The loser of any game switches to his opponent's choice next time, whilst the winner sticks with the same choice.

Suppose that games take place as a Poisson process of rate λ, and that, in each game, the pair of players is chosen randomly from the N children in the playground. Write down a Markov chain model for the proportions of children choosing paper, scissors and stone.

Show that, if λ is taken to be a suitable function of N, then, in the limit as $N \to \infty$, the Markov chain is well approximated (in a sense you should make precise) by the system of differential equations

$$
\begin{aligned}
\dot{x}_t^1 &= x_t^1(x_t^3 - x_t^2), \\
\dot{x}_t^2 &= x_t^2(x_t^1 - x_t^3), \\
\dot{x}_t^3 &= x_t^3(x_t^2 - x_t^1).
\end{aligned}
$$

[*Any appeal that you make to standard results should be supported by a precise statement, but not a proof.*]

Show that $x_t^1 x_t^2 x_t^3$ is a constant and comment on this fact in the light of the long time behaviour of the approximating Markov chains.