

M. PHIL. IN STATISTICAL SCIENCE

Monday 4 June 2001 9 to 11

MATHEMATICAL MODELS IN FINANCIAL MANAGEMENT

*Attempt any **THREE** questions. The questions carry equal weight.*

You may not start to read the questions
printed on the subsequent pages until
instructed to do so by the Invigilator.

1 An investor can invest an initial wealth of T pounds in one of two assets, bank deposits or equity, and is not allowed to borrow for this purpose. Assume that bank deposits do not involve any risk and offer a net return r for each pound invested. Equity holdings are risky. In the good state, which occurs with probability p , equity offers a return g per pound. In the bad state, which occurs with probability $1 - p$, the return is b . Assume that $b < r < g$ and $pg + (1 - p)b > r$. The investor is an expected utility maximizer, with the von Neumann–Morgenstern utility function in C^2 given by $u(w)$, where w denotes final wealth.

- (a) Suppose the investor holds an amount x in bank deposits and the rest of initial wealth in equity. Given this portfolio choice, find expressions for final wealth in the good state (call it w_g) and in the bad state (call it w_b). What is the expected utility of this portfolio?
- (b) What is meant by the term *risk neutral*? If this investor were risk neutral, what would the optimal portfolio choice be?
- (c) What is meant by the term *risk aversion*? Assuming the investor is risk averse, derive the first-order condition for optimal portfolio choice.
- (d) Assume now that the utility function is given by $u(w) = \ln w$ where w is final wealth and p satisfies

$$p > \left[1 + \frac{(g - r)(1 + r)}{(r - b)(1 + g)} \right]^{-1}.$$

What is the optimal portfolio?

- 2 (a) Assume that all asset rates of return in a market satisfy the one-factor model

$$\mathbf{r}_i = a_i + b_i \mathbf{f} \quad i = 1, \dots, n.$$

Show that in the absence of arbitrage opportunities

$$\bar{r}_i := \mathbb{E} \mathbf{r}_i = \lambda_0 + b_i \lambda_i \quad i = 1, \dots, n,$$

for suitable constants $\lambda_0, \lambda_1, \dots, \lambda_n$.

- (b) What is the relationship of this result to the CAPM?
 (c) Extend the result of (a) to m factors and interpret the λ_j s in financial terms.
 (d) Now consider the more realistic model

$$\mathbf{r}_i = a_i + \sum_{j=1}^m b_{ij} \mathbf{f}_j + \boldsymbol{\epsilon}_i \quad i = 1, \dots, n,$$

where $\mathbb{E} \boldsymbol{\epsilon}_i = 0$ and $\mathbb{E} \boldsymbol{\epsilon}_i^2 = \sigma_i^2 < s^2$ for all i . Suppose also that a portfolio of these n assets with weights w_i is *well-diversified* in the sense that $w_i < \frac{W}{n}$ for some constant $W \approx 1$. Show that the portfolio return satisfies

$$\mathbf{r} \approx a + \sum_{j=1}^m b_j \mathbf{f}_j$$

asymptotically for suitable constants a, b_1, \dots, b_m as the number of assets in the market $n \rightarrow \infty$ and discuss the implications of this result.

- 3 (a) Explain precisely what is meant by saying that a security price process S follows a geometric Brownian motion.
- (b) Describe briefly another model for such a price process S and indicate whether or not you think it is superior and why.
- (c) State the differential form of Ito's lemma and use a discrete time approximation to the underlying diffusion to explain the result heuristically.
- (d) Give a heuristic justification for the Black-Scholes partial differential equation for the value f of derivative security:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS \frac{\partial f}{\partial S} - rf + \frac{\partial f}{\partial t} = 0.$$

State precisely the assumptions underlying your derivation and define all parameters involved.

- (e) Explain the use of boundary conditions in solving the Black-Scholes PDE for European put and call option values. What is meant by put-call parity?
- (f) How is this equation used to value American put and call options?
- 4 (a) If F has inverse F^{-1} , then for a uniformly distributed sequence, $\xi_0, \xi_1, \xi_2, \dots$, show that elements of the sequence $\{\eta_i\} := \{F^{-1}(\xi_i)\}$ satisfy the distribution F .
- (b) Given a set of uniformly distributed pseudo-random numbers, how would one derive a set of normally distributed random numbers?
- (c) Briefly explain three variance reduction techniques, why they help to improve the accuracy of Monte Carlo estimates, the circumstances under which each is appropriate and their relative effectiveness.

- 5 (a) In the single factor *Heath-Jarrow-Morton (HJM)* model, given an initial forward rate curve $f(0, \cdot)$, the forward rate for each maturity T evolves as

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)d\mathbf{W}_t \quad 0 \leq t \leq T,$$

where \mathbf{W} is a Wiener process and α_t and σ_t can depend on \mathbf{W} and $f(\cdot, T)$, $T \geq t$, up to time t . Give expressions for $\mathbf{P}(t, T)$ and $r_t := f(t, t)$ and, defining the *cash bond* $\mathbf{B}_t = \exp(\int_0^t r_s ds)$, for the *discounted bond* $\mathbf{Z}_t(t, T) := \mathbf{B}_t^{-1}\mathbf{P}(t, T)$.

- (b) Defining $\Sigma(t, T) := -\int_t^T \sigma(t, u)du$, under the risk neutral measure Q , \mathbf{P} has dynamics

$$d\mathbf{P}(t, T) = \mathbf{P}(t, T) [r_t dt + \Sigma(t, T)d\mathbf{W}_t].$$

If \mathbf{X} is the single payoff of a derivative maturing at $T > t$ show that its current price at t is given by

$$\mathbf{V}(t, \mathbf{r}) = \mathbb{E}_Q \left[e^{-\int_t^T r_s ds} \mathbf{X} | \mathcal{F}_t \right].$$

- (c) Give an expression for the dynamics of the short rate r under Q . Is it necessarily Markov? Explain your answer.