UNIVERSITY OF CAMBRIDGE

M. PHIL. IN STATISTICAL SCIENCE

Thursday 7 June 2001 1.30 to 4.30

STATISTICAL THEORY

You should attempt ***FOUR*** *questions, no more than* **two** *of which should be from Section B.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Let a d-dimensional parameter vector θ be partitioned as $\theta = (\psi, \lambda)$.

Explain what is meant by *orthogonality* of ψ and λ.

Discuss briefly the consequences of parameter orthogonality for maximum likelihood estimation.

Suppose that Y is distributed according to a density of the form

$$P_Y(y;\theta) = a(\lambda, y) exp\{\lambda t(y;\psi)\}.$$

Show that ψ and λ are orthogonal.

2 Write a brief account of the concept and properties of *profile likelihood.*

Define what is meant by *modified profile likelihood.*

Let $Y_1, \ldots, Y_n$ be independent, identically distributed according to an inverse Gaussian distribution with density

$$\{\psi/(2\pi y^3)\}^{1/2} \; exp \; \{-\frac{\psi}{2\lambda^2 y} \; (y-\lambda)^2\}, \quad y > 0$$

where $\psi > 0$ and $\lambda > 0$. The parameter of interest is ψ.

Find the form of the profile log-likelihood function and of the modified profile log-likelihood.

3 (i) Let $Y_1, \ldots, Y_n$ be independent, identically distributed random variables with density $f_Y(y)$ and cumulant generating function $K_Y(t)$.

Describe in detail the *saddlepoint approximation* to the density of

$$\overline{Y} = n^{-1} \sum_{i=1}^{n} Y_i.$$

(ii) Let $Y_1, \ldots, Y_n$ be independent random variables each with a Laplace density

$$f_Y(y) = \exp\{-|y|\}/2, \; -\infty < y < \infty.$$

Show that the cumulant generating function is $K_Y(t) = -\log(1-t^2), \; |t| < 1$, and derive the form of the saddlepoint approximation to the density of $\overline{Y}$.

4 Explain what is meant by an *M-estimator* of a parameter θ, based on a given ψ function. Show that the influence function is proportional to ψ and derive an expression for the asymptotic variance of the M-estimator at a distribution F.

A location model on $\mathbb{R}$, with parameter space $\mathbb{R}$, is given by $F_\theta(x) = F(x-\theta)$, and an M-estimator is to be constructed using a ψ function of the form $\psi(x,\theta) = \psi(x-\theta)$. Let $IF(x;\psi,F)$ and $V(\psi,F)$ denote the influence function and asymptotic variance, respectively, at a distribution F, and let Φ denote the standard normal distribution. Show that the problem of choosing an odd, non-decreasing ψ function which minimises $V(\psi,\Phi)$ among all estimators with

$$|IF(x;\psi,\Phi)| \leqslant C < \infty,$$

for given $C \geqslant \sqrt{\pi/2}$, is solved by

$$\psi(x) = \max\{-K, \min\{x, K\}\},$$

with $C = K/\{2\Phi(K) - 1\}$.

5 (i) What is meant by a *maximal invariant statistic* with respect to a group of transformations on a sample space?

State and prove a result which establishes the importance of maximal invariants in the construction of non-parametric tests.

(ii) Let $X_1, \ldots, X_n$ be independent, identically distributed with continuous distribution function F_X, and $Y_1, \ldots, Y_m$ be independent, identically distributed from a continuous distribution function F_Y.

Describe the *Wilcoxon test* of $H_0 : F_X(z) = F_Y(z), \forall z$, against $H_1 : F_X(z) \geqslant F_Y(z), \forall z$, and justify the test in terms of the discussion in (i) above.

State, without proof, the asymptotic null distribution of the Wilcoxon test statistic.

6 Write an account of *one* of the following:

(i) Edgeworth and Laplace approximations;

(ii) The p^*-formula for the density of the maximum likelihood estimator;

(iii) Exponential families and transformation models.

SECTION B

7 (i) Assume that the n-dimensional observation vector Y may be written

$$\Omega : Y = X\beta + \epsilon$$

where X is a given $n \times p$ matrix of rank p, β is an unknown vector, and

$$\epsilon \sim N_n(0, \sigma^2 I).$$

Let $Q(\beta) = (Y - X\beta)^T(Y - X\beta)$. Show that $Q(\beta)$ is a convex function of β, and find $\hat{\beta}$, the least-squares estimator of β. Show also that

$$Q(\hat{\beta}) = Y^T(I - H)Y$$

where H is a matrix that you should define.

(ii) Let $\hat{\epsilon} = Y - X\hat{\beta}$. Find the distribution of $\hat{\epsilon}$, and discuss how this may be used to perform diagnostic checks of Ω.

(iii) Suppose that your data actually corresponded to the model

$$Y_i \sim N(\mu_i, \sigma_i^2),\ 1 \le i \le n, \text{with } \sigma_i^2 \propto \mu_i^2.$$

How would your diagnostic checks detect this, and what transformation of Y_i would be appropriate?

8 Suppose that $Y_1, \cdots, Y_n$ are independent Poisson random variables, with $\mathbb{E}(Y_i) = \mu_i t_i,\ 1 \le i \le n$, where $t_1, \cdots, t_n$ are given times. Discuss carefully how to fit the model

$$H_0 : \log\mu_i = \beta^T x_i,\ 1 \le i \le n,$$

where $x_1, \cdots, x_n$ are given covariates, and β is a vector of unknown parameters.

9 Write a brief account of the role of *conditioning* in classical statistical inference.

Contrast briefly the handling of nuisance parameters in classical approaches to inference with that in the Bayesian approach.

10 Let $X_1, \ldots, X_n$ be independent, identically distributed random variables, with the exponential density $f(x;\theta) = \theta e^{-\theta x}, x > 0$.

Obtain the maximum likelihood estimator $\hat{\theta}$ of θ. What is the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$?

Show that $\hat{\theta}$ is biased as an estimator of θ.

What is the minimum variance unbiased estimator of θ? Justify your answer carefully, stating clearly any results you use.