Algebraic Geometry Part III Catch-up Workshop 2015

Jack Smith

June 6, 2016

1 Abstract

Algebraic Geometry

This workshop will give a basic introduction to affine algebraic geometry, assuming no prior exposure to the subject. In particular, we will cover:

- Affine space and algebraic sets
- The Hilbert basis theorem and applications
- The Zariski topology on affine space
- Irreducibility and affine varieties
- The Nullstellensatz
- Morphisms of affine varieties.

If there's time we may also touch on projective varieties.

What we expect you to know

- Elementary point-set topology: topological spaces, continuity, closure of a subset etc
- Commutative algebra, at roughly the level covered in the Rings and Modules workshop: rings, ideals (including prime and maximal) and quotients, algebras over fields (in particular, some familiarity with polynomial rings over fields).

Useful for Part III courses

Algebraic Geometry, Commutative Algebra, Elliptic Curves

2 Talk

2.1 Preliminaries

Useful resources:

- Hartshorne 'Algebraic Geometry' (classic textbook, on which I think this year's course is based, although it's quite dense; I'll mainly try to match terminology and notation with Chapter 1 of this book).
- Ravi Vakil's online notes 'Math 216: Foundations of Algebraic Geometry'.
- Eisenbud 'Commutative Algebra with a view toward algebraic geometry' (covers all the algebra you might need, with a geometric flavour—it has pictures).

- Pelham Wilson's online notes for the 'Preliminary Chapter 0' of his Part III Algebraic Geometry course from last year cover much of this catch-up material but are pretty brief (warning: this year's course has a different lecturer so will be different).

Notation: k is an algebraically closed field—can think $k = \mathbb{C}$ if you prefer; all rings are commutative with 1; $U \subset V$ means U is a subset of V, but does not exclude $U = V$.

2.2 Overview

In geometry we study spaces with various different kinds of structures. We could start with studying just sets; these are quite un-geometric. Introducing a notion of continuity we get topological spaces, which are much more interesting. In differential geometry we introduce a notion of smoothness and get manifolds. In algebraic geometry we introduce a notion of polynomial-ness and get varieties (or more generally schemes). Just as a manifold is a topological space with some extra structure (and a toplogical space is itself a set with some extra structure), a variety is a topological space together with some extra structure, which comes in the form of a sheaf of local regular ('polynomial') functions.

We won't define varieties in general today, but we will discuss the local models: affine varieties. Just as manifolds locally look like $\mathbb{R}^n$, a general variety locally looks like affine varieties. We won't really talk about sheaves either.

2.3 Algebraic sets

Definition: *Affine n-space* over k, denoted by $\mathbb{A}^n$, is a variety whose underlying set is k^n—so it's k^n equipped with a particular topology (which we'll define later) and a particular sheaf of regular functions (which we won't). A point $p \in \mathbb{A}^n$ is represented by an n-tuple $(a_1, \ldots, a_n)$, the *coordinates* of p.

The set of *regular functions* on $\mathbb{A}^n$ is the polynomial ring $A = k[X_1, \ldots, X_n]$. The polynomial $f(X_1, \ldots, X_n)$ evaluates at p to $f(a_1, \ldots, a_n)$. For a subset $T \subset A$ we define the zero set $Z(T) \subset A$ of T to be the set of common zeros of the polynomials in T. When n is small, we'll typically call our variables things like X and Y rather than X_1 and X_2.

Examples: In $\mathbb{A}^2$ with regular functions $k[X, Y]$:

- $Z(Y) = \{Y = 0\}$ is a line—the X-axis (draw picture). This picture is drawn with $k = \mathbb{R}$, which is not algebraically closed, but it's a useful way to visualise objects—obviously can't draw a diagram in $\mathbb{C}^2$. Over other fields pictures break down a bit but they're good to keep in mind for intuition.
- $Z(XY - 1) = \{XY = 1\}$ is a hyperbola (draw picture).
- Warning about pictures over $\mathbb{R}$: some spaces obviously can't be drawn, e.g. $Z(X - i)$, whilst others look empty but aren't, e.g. $Z(X^2 + 1)$.

Definition: An *algebraic set* $Y \subset \mathbb{A}^n$ is a subset of the form $Z(T)$ for some $T \subset A$.

Look at questions 1 and 2.

2.4 The Hilbert basis theorem

If we have sets $T_1 \subset T_2 \subset A$ and a point $p \in Z(T_2)$ then every function in T_2 vanishes at p. So every function in T_1 vanishes at p, and thus $p \in Z(T_1)$. In other words, if $T_1 \subset T_2$ then $Z(T_2) \subset Z(T_1)$.

Now consider a set $T \subset A$ and let $J \subset A$ be the ideal generated by T. Have $T \subset J$ so $Z(J) \subset Z(T)$.

Claim: $Z(T) \subset Z(J)$ so $Z(T) = Z(J)$.

Proof: Suppose $p \in Z(T)$ and $f \in J$. Need $f(p) = 0$. Can write $f = a_1 t_1 + \cdots + a_m t_m$ for some $t_i \in T$ and $a_i \in A$. Since $p \in Z(T)$ have $t_i(p) = 0$ for all i. So

$$f(p) = a_1(p)t_1(p) + \cdots + a_m(p)t_m(p) = 0. \qquad \square$$

So given an algebraic set $Z(T)$ we may as well replace T by the ideal J that it generates.

Recall, a ring R is Noetherian if every ideal is finitely generated, or equivalently if every ascending chain of ideals terminates, i.e. if

$$J_1 \subset J_2 \subset J_3 \subset \dots$$

is a nested sequence of ideals in R then there exists N such that $J_n = J_N$ for all $n \geq N$.

Examples:

- $\mathbb{Z}$ is Noetherian—it's a principal ideal domain so all ideals are finitely generated.
- Any field k is Noetherian—there are only two ideals, (0) and k, so any chain terminates (also both ideals are finitely generated).

Hilbert basis theorem: If R is a Noetherian ring then $R[X]$ is also Noetherian.

There's an exercise at the end of the question sheet that guides you through a proof of this.

Corollary: $A = k[X_1, \dots, X_n]$ is Noetherian for all n.

Proof: k is Noetherian, so HBT implies $k[X_1]$ is Noetherian. Applying HBT again we see that $k[X_1][X_2]$ is Noetherian, but this ring is just $k[X_1, X_2]$. Induct. $\square$

Corollary: Any algebraic set $Z(T)$ can be written as the zero set of a finite collection of polynomials.

Proof: Let T generate the ideal J in A. Since A is Noetherian, we can pick a finite collection of generators $T' = \{f_1, \dots, f_m\}$ for J. Then $Z(T) = Z(J) = Z(T')$. $\square$

Look at question 3.

2.5 The Zariski topology

The algebraic sets form the closed sets of a topology on $\mathbb{A}^n$, the *Zariski topology*. Need to check:

- $\mathbb{A}^n$ is closed—it's $Z(0)$.
- $\emptyset$ is closed—it's $Z(1)$.
- Finite unions: given closed sets $Y_1, \dots, Y_m$, we need $\bigcup_{i=1}^m Y_i$ closed. We can write $Y_i = Z(J_i)$ for some ideals $J_1, \dots, J_m$, and then:

  Claim: $\bigcup_{i=1}^m Z(J_i) = Z\big(\prod_{i=1}^m J_i\big)$ (recall $\prod J_i$ is the ideal generated by products $f_1 \dots f_m$ with $f_i \in J_i$).

  Sketch proof: If $p \in Z(J_l)$ then every function $f_l \in J_l$ vanishes at p, so every product $f_1 \dots f_m$ vanishes at p. Hence $p \in Z\big(\prod J_i\big)$. So $Z(J_l) \subset Z\big(\prod J_i\big)$ for all l.

  Conversely suppose $p \notin \bigcup Z(J_i)$. Then for each i there exists a function $f_i \in J_i$ not vanishing at p. So $f_1 \dots f_m$ doesn't vanish at p. Hence $p \notin Z\big(\prod J_i\big)$. $\square$
- Arbitrary intersections: given closed sets $Z(J_\alpha)$ for ideals J_α, $\alpha \in \mathcal{A}$, need $\bigcap_{\alpha \in \mathcal{A}} Z(J_\alpha)$ closed. In fact it's $Z\big(\sum_{\alpha \in \mathcal{A}} J_\alpha\big)$ (recall $\sum_{\alpha \in \mathcal{A}} J_\alpha$ is the ideal generated by all of the J_α).

This is the topology we use on the variety $\mathbb{A}^n$.

Examples:

- Singleton sets are closed: if $p \in \mathbb{A}^n$ has coordinates $(a_1, \dots, a_n)$ then

  $$\{p\} = Z(X_1 - a_1, \dots, X_n - a_n).$$

  Taking finite unions, we see that finite sets are closed.

- The Zariski topology is compact but not Hausdorff, so when $k = \mathbb{C}$ it is very unlike the usual topology on $\mathbb{C}^n$.

We've just seen that finite subsets of $\mathbb{A}^1$ are closed. The converse can't quite be true since $\mathbb{A}^1$ is infinite but must be closed. But it's almost true.

Claim: The proper closed subsets of $\mathbb{A}^1$ are finite.

Proof: Let $Y \subset \mathbb{A}^1$ be a proper closed set. Thus $Y = Z(J)$ for some ideal $J \subset k[X]$. This ring is a principal ideal domain so $J = (f)$ for some polynomial f, which must be non-zero (otherwise Y would be all of $\mathbb{A}^1$). So $Y = Z(f)$ is just the set of roots of f. And non-zero polynomials over a field can have only finitely many roots. $\square$

So we have a topology on $\mathbb{A}^n$ and a k-algebra of regular functions A from $\mathbb{A}^n$ to k. Identifying k with $\mathbb{A}^1$ it makes sense to ask if the regular functions are continuous.

Claim: The regular functions on $\mathbb{A}^n$ are continuous as maps $\mathbb{A}^n \to \mathbb{A}^1$.

Sketch proof: Since proper closed subsets of $\mathbb{A}^1$ are finite, it is sufficient to prove that if $f \in A = k[X_1, \ldots, X_n]$ is a regular function and $a \in k$ then the set $f^{-1}(a)$ is closed in $\mathbb{A}^n$. But this set is precisely the zero set of $f - a$, i.e. it is $Z(f - a)$, which is closed. $\square$

Now look at questions 4 and 5.

2.6 Affine varieties

Given a Zariski-closed subset $Y \subset \mathbb{A}^n$ we can view it as a topological space with the subspace topology. This is called the Zariski topology on Y. There is a natural way to put a sheaf of regular functions on Y coming from $\mathbb{A}^n$.

Definition: An *affine variety* is a variety formed in this way from a Zariski-closed subset of affine space.

Earlier we defined the map

$$\{\text{subsets of } A\} \xrightarrow{Z} \{\text{subsets of } \mathbb{A}^n\}.$$

The image consists of the Zariski-closed subsets, and we may restrict the domain to ideals without changing this.

Now we define a map the other way: given any subset $Y \subset \mathbb{A}^n$, let the *ideal of* Y, $I(Y)$, be the set of polynomials which vanish at every point of Y:

$$I(Y) = \{f \in A : f(p) = 0 \text{ for all } p \in Y\}.$$

This *is* an ideal in A.

So we get

$$\{\text{subsets of } A\} \xleftarrow{I} \{\text{subsets of } \mathbb{A}^n\}.$$

Note that if $Y_1 \subset Y_2$ are subsets of $\mathbb{A}^n$ then any polynomial vanishing at every point of Y_2 vanishes at every point of Y_1, i.e. $I(Y_2) \subset I(Y_1)$. The composition $Z \circ I$ maps a set to its closure in the Zariski topology (see Question 6).

For an affine variety $Y \subset \mathbb{A}^n$, we can restrict functions on $\mathbb{A}^n$, i.e. elements of $A = k[X_1, \ldots, X_n]$, to get functions on Y. This restriction map is a k-algebra homomorphism into the ring of continuous functions on Y, whose kernel is precisely the functions vanishing on Y, i.e. $I(Y)$.

Definition: The *regular functions* on Y are the functions in the image of this homomorphism. The *coordinate ring* $A(Y)$ of Y is the ring of regular functions on Y. By the first isomorphism theorem, we can think of this as $A/I(Y)$. Warning: it is not in general true that $A(Z(J)) = A/J$; we have to work with $A/I(Z(J))$.

Examples:

- $A(\mathbb{A}^n) = k[X_1, \ldots, X_n]$.
- For the parabola $Y = Z(X_2 - X_1^2)$ we have $I(Y) = (X_2 - X_1^2)$ so $A(Y) = k[X_1, X_2]/(X_2 - X_1^2) \cong k[X]$ (apply the first isomorphism theorem to the map $X_1 \mapsto X$, $X_2 \mapsto X^2$).

Look at next block of questions.

2.7 Irreducibility

Definition: Let X be a topological space. X is *irreducible* if it's non-empty and whenever $X = E \cup F$ for closed sets E, F we have $X = E$ or $X = F$. A subset $Y \subset X$ is irreducible if it's non-empty and whenever $Y \subset E \cup F$ for closed sets E and F we have $Y \subset E$ or $Y \subset F$. This is equivalent to Y being irreducible as a space equipped with the subspace topology from X. A space/subset is *reducible* if it's not irreducible. Irreducibility is like a stronger version of connectedness.

Examples:

- $\mathbb{R}$ with the standard Euclidean topology is reducible; we can take $E = (-\infty, 0]$ and $F = [0, \infty)$.
- $\mathbb{A}^1$ is irreducible: $\mathbb{A}^1$ is infinite (since k is algebraically closed), so if $\mathbb{A}^1 = E \cup F$ for closed sets E and F then E or F is infinite, hence E or F is all of $\mathbb{A}^1$.
- In $\mathbb{A}^2$ the set $Z(XY)$ is reducible: take $E = Z(X)$ and $F = Z(Y)$ (draw picture).

Definition: An afffine variety is *irreducible* if its underlying topological space is irreducible. (Warning: in Hartshorne all varieties are assumed irreducible.)

Claim: An affine variety $V \subset \mathbb{A}^n$ can be written as a finite union of irreducible affine varieties $V_1, \ldots, V_n$ with $V_i \not\subseteq V_j$ for all distinct i and j (the V_i are unique up to reordering and are called the *components* of V).

Sketch proof: If V is irreducible then we're done. If not we can write $V = V_1 \cup V_2$ for proper closed subsets V' and V''. Each of these is irreducible or can be decomposed again. Keep going. We need to show this process terminates. So suppose for contradiction that we have a strictly descending sequence of closed subsets

$$V \supsetneq V_1 \supsetneq V_2 \supsetneq \ldots.$$

Then we get an ascending chain of ideals in A

$$I(V) \subset I(V_1) \subset V_2 \subset \ldots.$$

And the inclusions are all strict: if $I(V_i) = I(V_j)$ then $V_i = Z(I(V_i)) = Z(I(V_j)) = V_j$. This contradicts the Hilbert basis theorem.

Therefore the process *does* terminate and we can write $V = V_1 \cup \cdots \cup V_n$ for irreducible closed subsets V_i. If $V_i \subset V_j$ for some $i \neq j$ then throw out V_i. Keep applying this until no V_i is contained in any other. We then get the result. $\square$

Examples:

- The components of the variety $Z(XY) \subset \mathbb{A}^2$ (draw picture) are the two lines (each of these *is* irreducible by the same argument as for $\mathbb{A}^1$; in fact the lines are isomorphic to $\mathbb{A}^1$, which we'll define later).
- The hyperbola $Z(XY - 1)$ (draw picture) is irreducible (we'll prove this later) although it looks like it has two separate components.

Look at the next block of questions.

2.8 The Nullstellensatz

Recall I and Z. We've seen what $Z \circ I$ does. What about $I \circ Z$?

Definition: For an ideal J in a ring R, the *radical* of J is

$$\sqrt{J} \coloneqq \{r \in R : r^n \in J \text{ for some positive integer } n\}.$$

This is an ideal of R. Say J is *radical* if $J = \sqrt{J}$.

Example: If J is a prime ideal then it is radical: if $r \in \sqrt{J}$ then for some positive integer n we have $r^n \in J$, so by primality we have $r \in J$. Hence $\sqrt{J} \subset J$. The other inclusion is trivial.

Nullstellensatz: For an algebraically closed field k, and an ideal $J \subset A = k[X_1, \ldots, X_n]$, we have $I(Z(J)) = \sqrt{J}$.

Examples: In $\mathbb{A}^n$ with coordinate ring $A = k[X_1, \ldots, X_n]$:

- If $J = (X_1^2) \subset A$ then $Z(J)$ is the linear subspace $\{X_1 = 0\}$ so $I(Z(J)) = (X_1) = \sqrt{(X_1^2)}$
- Non-example: if $k = \mathbb{R}$ and $J = (X_1^2 + 1)$ then $Z(J) = \emptyset$ but $\sqrt{J} \neq I(\emptyset) = A$; the problem is that $\mathbb{R}$ is not algebraically closed

So we get a bijective inclusion-reversing correspondence

$$\{\text{radical ideals in } A\} \longleftrightarrow \{\text{Zariski-closed sets in } \mathbb{A}^n\}.$$

Claim: The ideals corresponding to irreducible sets are precisely the primes. This is an example of the interaction between algebra and topology.

Sketch proof: Let $Y \subset \mathbb{A}^n$ be closed. If $I(Y)$ is prime and $Y = E \cup F$ for closed subsets E and F (closed in Y and, equivalently, in X) then $I(Y) = I(E) \cap I(F) \supset I(E)I(F)$. Since $I(Y)$ is prime, we deduce that $I(Y) \supset I(E)$ or $I(Y) \supset I(F)$, so $Y = Z(I(Y)) \subset Z(I(E)) = E$ or $Z \subset F$. Hence Y is irreducible.

Conversely, if $I(Y)$ is not prime then pick $f, g \in A$ with $fg \in I(Y)$ but $f, g \notin I(Y)$. We get an ideal $(fg) \subset I(Y)$, so $Y = Z(I(Y)) \subset Z(fg) = Z(f) \cup Z(g)$. But $Y \not\subseteq Z(f), Z(g)$ since $f, g \notin I(Y)$. Setting $E = Z(f)$ and $F = Z(g)$ we see that Y is not irreducible. $\square$

We thus have

$$Y \text{ irreducible} \iff I(Y) \text{ prime} \iff A(Y) \text{ an integral domain.}$$

Examples:

- $\mathbb{A}^n$ has coordinate ring $k[X_1, \ldots, X_n]$ which is an integral domain, so is irreducible.
- We saw that $Z(XY) \subset \mathbb{A}^2$ is reducible. The coordinate ring is $k[X, Y]/(XY)$, which is not an integral domain, since it contains the non-zero functions X and Y whose product is 0. The individual lines $Z(X)$ and $Z(Y)$ are irreducible since $I(Z(X)) = (X)$ and $I(Z(Y)) = (Y)$ and these ideals are obviously prime.
- The hyperbola $Z(XY - 1) \in \mathbb{A}^2$ is irreducible because

  $$k[X, Y]/(XY - 1) \cong k[X, X^{-1}]$$

  and the right-hand side is an integral domain.

The maps I and Z can be applied to subsets of a variety Y and subsets of its coordinate ring $A(Y)$ to get

$$\{\text{radical ideals in } A(Y)\} \longleftrightarrow \{\text{closed subsets of } Y\}$$

Again irreducible subsets correspond to prime ideals.

Maximal ideals should correspond to minimal closed subsets, i.e. points. This is indeed the case:

- For a point $p \in Y$ we have a surjective k-algebra homomorphism $\mathrm{ev_p} : A(Y) \to k$ given by $f \mapsto f(p)$ (it's surjective because of the constant functions in $A(Y)$). Since the image is a field, the kernel is maximal ideal $\mathfrak{m} \subset A(Y)$. This is precisely $I(p)$.

- Conversely, given a maximal ideal $\mathfrak{m} \subset A(Y)$ we have a closed subvariety $Z(\mathfrak{m}) \subset Y$ with

$$I(Z(\mathfrak{m})) = \sqrt{\mathfrak{m}} = \mathfrak{m} \neq A(Y)$$

so $Z(\mathfrak{m}) \neq \emptyset$. If $Z(\mathfrak{m})$ contains a point p then $\{p\}$ is a closed subvariety of $Z(\mathfrak{m})$ so $I(p)$ is a radical ideal in $A(Y)$ containing $\mathfrak{m}$—by maximality of $\mathfrak{m}$ we see that $I(p) = \mathfrak{m}$ and so $Z(\mathfrak{m}) = p$.

Therefore Z and I give bijective inclusion-reversing correspondences

$$\begin{aligned}\{\text{radical ideals in } A(Y)\} &\longleftrightarrow \{\text{closed subsets of } Y\}\\ \{\text{prime ideals in } A(Y)\} &\longleftrightarrow \{\text{irreducible closed subsets of } Y\}\\ \{\text{maximal ideals in } A(Y)\} &\longleftrightarrow \{\text{points of } Y\}\end{aligned}$$

Now look at next block of questions.

2.9 Morphisms

We now have objects—affine varieties. What is the right notion of maps between them?

Definition: For an affine variety $Y \subset \mathbb{A}^n$, a *morphism* $Y \to \mathbb{A}^m$ is a continuous map given in components by $(f_1, \dots, f_m)$ for some regular functions $f_1, \dots, f_m \in A(Y)$ (so it's basically given by polynomials). If $Y' \subset \mathbb{A}^m$ is an affine variety, a morphism from Y to Y' is a morphism $Y \to \mathbb{A}^m$ whose image lies in Y'. Note that the identity map on a variety is a morphism, and the composition of two morphisms is a morphism. An *isomorphism* from Y to Y' is a morphism $\phi : Y \to Y'$ such that there exists a morphism $\psi : Y' \to Y$ with $\psi \circ \phi = \mathrm{id}_Y$ and $\phi \circ \psi = \mathrm{id}_{Y'}$.

Examples:

- The varieties $Y := \mathbb{A}^1$ and $Y' := Z(X_2 - X_1^2) \subset \mathbb{A}^2$ are isomorphic via

$$\phi : Y \to Y',\ t \mapsto (t, t^2)$$

and

$$\psi : Y' \to Y,\ (t, t^2) \mapsto t.$$

(Draw picture.)

- Isomorphic varieties are homeomorphic; the converse is very false.

Given affine varieties $Y \subset \mathbb{A}^n, Y' \subset \mathbb{A}^m$, a morphism $\phi : Y \to Y'$, and a continuous function f on Y', we can form a continuous function $f \circ \phi$ on Y. If f is regular (i.e. polynomial) then so is $f \circ \phi$.

Definition: This k-algebra homomorphism $A(Y') \to A(Y)$, $f \mapsto f \circ \phi$ is the *pullback by* ϕ, denoted by ϕ^*.

Example: Consider the morphism $\phi : \mathbb{A}^1 \to \mathbb{A}^2$ given by $t \mapsto (t, t^2)$. Then ϕ^* maps $k[X, Y] \to k[T]$. Note that I'm calling the variable on $\mathbb{A}^1$ T to avoid confusion. Then ϕ^* sends X to T and Y to T^2.

Conversely, given a k-algebra homomorphism $\alpha : A(Y') \to A(Y)$ we can build a morphism $\alpha^* : Y \to Y'$; I don't want to get into the details now but you can think about how to do this. These * operations have nice properties, e.g. $\phi^{**} = \phi$ and $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$.

Key fact: The operations * define a bijection

$$\{\text{morphisms } Y \to Y'\} \leftrightarrow \{k\text{-algebra homomorphisms } A(Y') \to A(Y)\}$$

Look at the last question.

2.10 Next steps

So far we have talked about affine varieties and the global regular functions defined on them (i.e. 'polynomial' functions which are defined on the whole variety). In order to build the full structure of a variety we need to define a sheaf of regular functions. In particular we need to understand what the correct definition of a regular function on an arbitrary open set is.

Example: On $\mathbb{A}^1$, regular functions are polynomials in a variable X. The function $1/X$ is perfectly good at most points but has a singularity at the origin. We'll say $1/X$ is regular on the open subset $\{X \neq 0\}$ (this is open since its complement is the closed set $Z(X)$). If $1/X$ is regular then $1/X^r$ had better be regular for all non-negative integers r. And then g/X^r had better be regular for all r and all polynomials g.

More generally, for a non-zero polynomial $f \in A = k[X_1, \dots, X_n]$ on $\mathbb{A}^n$, we define the open set $D(f)$ to be the $\{p \in \mathbb{A}^n : f(p) \neq 0\}$ and then set the regular functions on $D(f)$ to be the ring

$$A_f := \left\{ \frac{g}{f^r} : g \in A, r \in \mathbb{Z}_{\geq 0} \right\},$$

a subring of the field of rational functions $k(X_1, \dots, X_n)$.

More generally still, for an *irreducible* affine variety Y, with coordinate ring $A(Y)$, and a non-zero function $f \in A(Y)$, we get an open set $D(f)$ in Y defined by $\{p \in Y : f(p) \neq 0\}$, and we define the regular functions on $D(f)$ to be

$$A(Y)_f := \left\{ \frac{g}{f^r} : g \in A(Y), r \in \mathbb{Z}_{\geq 0} \right\}.$$

This is a subring of the *function field* $K(Y)$ of Y, which is defined to be the field of fractions of $A(Y)$.

If Y is not irreducible, we can't build a build a field of fractions and we have to be a little more careful. Need the notion of *localisation*.

Key fact: The sets $D(f)$ form a basis for the Zariski topology on any affine variety (not necessarily irreducible), i.e. any open set can be written as a union of sets of this form.

Proof: Take an open set U in an affine variety Y, and let p be a point of U. Write U as $Y \setminus Z(J)$ for an ideal J. We want to find an f such that

$$p \in D(f) \subset U,$$

in other words $f(p) \neq 0$ but f vanishes on $Z(J)$. Since p is not in $Z(J)$, there exists a function in J not vanishing at p. Take f to be such a function. □

It's easy to define regular functions on sets of the form $D(f)$, and because of this fact we can then use sheafification to construct the whole sheaf of regular functions.