UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday, 31 May, 2012 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1C Vectors and Matrices

(a) Let R be the set of all $z \in \mathbb{C}$ with real part 1. Draw a picture of R and the image of R under the map $z \mapsto e^z$ in the complex plane.

(b) For each of the following equations, find all complex numbers z which satisfy it:

(i) $e^z = e$,

(ii) $(\log z)^2 = -\dfrac{\pi^2}{4}$.

(c) Prove that there is no complex number z satisfying $|z| - z = i$.

2A Vectors and Matrices

Define what is meant by the terms *rotation*, *reflection*, *dilation* and *shear*. Give examples of real 2×2 matrices representing each of these.

Consider the three 2×2 matrices

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad C = AB\,.$$

Identify the three matrices in terms of your definitions above.

3E Analysis I

What does it mean to say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *continuous* at $x_0 \in \mathbb{R}$?

Give an example of a continuous function $f\colon (0,1] \to \mathbb{R}$ which is bounded but attains neither its upper bound nor its lower bound.

The function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and non-negative, and satisfies $f(x) \to 0$ as $x \to \infty$ and $f(x) \to 0$ as $x \to -\infty$. Show that f is bounded above and attains its upper bound.

[Standard results about continuous functions on closed bounded intervals may be used without proof if clearly stated.]

4F **Analysis I**

Let $f, g : [0,1] \to \mathbb{R}$ be continuous functions with $g(x) \geqslant 0$ for $x \in [0,1]$. Show that

$$\int_0^1 f(x)g(x)\,dx \leqslant M \int_0^1 g(x)\,dx\,,$$

where $M = \sup\{|f(x)| : x \in [0,1]\}$.

Prove there exists $\alpha \in [0,1]$ such that

$$\int_0^1 f(x)g(x)\,dx = f(\alpha)\int_0^1 g(x)\,dx\,.$$

[Standard results about continuous functions and their integrals may be used without proof, if clearly stated.]

SECTION II

5C Vectors and Matrices

The equation of a plane Π in $\mathbb{R}^3$ is

$$\mathbf{x} \cdot \mathbf{n} = d\,,$$

where d is a constant scalar and $\mathbf{n}$ is a unit vector normal to Π. What is the distance of the plane from the origin O?

A sphere S with centre $\mathbf{p}$ and radius r satisfies the equation

$$|\mathbf{x} - \mathbf{p}|^2 = r^2\,.$$

Show that the intersection of Π and S contains exactly one point if $|\mathbf{p} \cdot \mathbf{n} - d| = r$.

The tetrahedron $OABC$ is defined by the vectors $\mathbf{a} = \vec{OA}$, $\mathbf{b} = \vec{OB}$, and $\mathbf{c} = \vec{OC}$ with $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) > 0$. What does the condition $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) > 0$ imply about the set of vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$? A sphere T_r with radius $r > 0$ lies inside the tetrahedron and intersects each of the three faces OAB, OBC, and OCA in exactly one point. Show that the centre P of T_r satisfies

$$\vec{OP} = r\frac{|\mathbf{b} \times \mathbf{c}|\mathbf{a} + |\mathbf{c} \times \mathbf{a}|\mathbf{b} + |\mathbf{a} \times \mathbf{b}|\mathbf{c}}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}\,.$$

Given that the vector $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$ is orthogonal to the plane Ψ of the face ABC, obtain an equation for Ψ. What is the distance of Ψ from the origin?

6A Vectors and Matrices

Explain why the number of solutions $\mathbf{x}$ of the simultaneous linear equations $A\mathbf{x} = \mathbf{b}$ is 0, 1 or infinity, where A is a real 3×3 matrix and $\mathbf{x}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^3$. State necessary and sufficient conditions on A and $\mathbf{b}$ for each of these possibilities to hold.

Let A and B be real 3×3 matrices. Give necessary and sufficient conditions on A for there to exist a unique real 3×3 matrix X satisfying $AX = B$.

Find X when

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & -1 \end{pmatrix}$$

7B Vectors and Matrices

(a) Consider the matrix

$$M = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}.$$

Determine whether or not M is diagonalisable.

(b) Prove that if A and B are similar matrices then A and B have the same eigenvalues with the same corresponding algebraic multiplicities.

Is the converse true? Give either a proof (if true) or a counterexample with a brief reason (if false).

(c) State the Cayley-Hamilton theorem for a complex matrix A and prove it in the case when A is a 2×2 diagonalisable matrix.

Suppose that an $n \times n$ matrix B has $B^k = \mathbf{0}$ for some $k > n$ (where $\mathbf{0}$ denotes the zero matrix). Show that $B^n = \mathbf{0}$.

8B Vectors and Matrices

(a) (i) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

(ii) Show that the quadric $\mathcal{Q}$ in $\mathbb{R}^3$ defined by

$$3x^2 + 2xy + 2y^2 + 2xz + 2z^2 = 1$$

is an ellipsoid. Find the matrix B of a linear transformation of $\mathbb{R}^3$ that will map $\mathcal{Q}$ onto the unit sphere $x^2 + y^2 + z^2 = 1$.

(b) Let P be a real orthogonal matrix. Prove that:

(i) as a mapping of vectors, P preserves inner products;

(ii) if λ is an eigenvalue of P then $|\lambda| = 1$ and λ^* is also an eigenvalue of P.

Now let Q be a real orthogonal 3×3 matrix having $\lambda = 1$ as an eigenvalue of algebraic multiplicity 2. Give a geometrical description of the action of Q on $\mathbb{R}^3$, giving a reason for your answer. [You may assume that orthogonal matrices are always diagonalisable.]

9E Analysis I

(a) What does it mean to say that the sequence (x_n) of real numbers *converges* to $\ell \in \mathbb{R}$?

Suppose that $(y_n^{(1)})$, $(y_n^{(2)})$,..., $(y_n^{(k)})$ are sequences of real numbers converging to the same limit ℓ. Let (x_n) be a sequence such that for every n,

$$x_n \in \{y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(k)}\}\,.$$

Show that (x_n) also converges to ℓ.

Find a collection of sequences $(y_n^{(j)})$, $j = 1, 2, \ldots$ such that for every j, $(y_n^{(j)}) \to \ell$ but the sequence (x_n) defined by $x_n = y_n^{(n)}$ diverges.

(b) Let a, b be real numbers with $0 < a < b$. Sequences (a_n), (b_n) are defined by $a_1 = a$, $b_1 = b$ and

$$\text{for all } n \geqslant 1, \quad a_{n+1} = \sqrt{a_n b_n}, \quad b_{n+1} = \frac{a_n + b_n}{2}\,.$$

Show that (a_n) and (b_n) converge to the same limit.

10D Analysis I

Let (a_n) be a sequence of reals.

(i) Show that if the sequence $(a_{n+1} - a_n)$ is convergent then so is the sequence $\left(\frac{a_n}{n}\right)$.

(ii) Give an example to show the sequence $\left(\frac{a_n}{n}\right)$ being convergent does not imply that the sequence $(a_{n+1} - a_n)$ is convergent.

(iii) If $a_{n+k} - a_n \to 0$ as $n \to \infty$ for each positive integer k, does it follow that (a_n) is convergent? Justify your answer.

(iv) If $a_{n+f(n)} - a_n \to 0$ as $n \to \infty$ for every function f from the positive integers to the positive integers, does it follow that (a_n) is convergent? Justify your answer.

11D Analysis I

Let f be a continuous function from $(0,1)$ to $(0,1)$ such that $f(x) < x$ for every $0 < x < 1$. We write f^n for the n-fold composition of f with itself (so for example $f^2(x) = f(f(x))$).

(i) Prove that for every $0 < x < 1$ we have $f^n(x) \to 0$ as $n \to \infty$.

(ii) Must it be the case that for every $\epsilon > 0$ there exists n with the property that $f^n(x) < \epsilon$ for all $0 < x < 1$? Justify your answer.

Now suppose that we remove the condition that f be continuous.

(iii) Give an example to show that it need not be the case that for every $0 < x < 1$ we have $f^n(x) \to 0$ as $n \to \infty$.

(iv) Must it be the case that for *some* $0 < x < 1$ we have $f^n(x) \to 0$ as $n \to \infty$? Justify your answer.

12F Analysis I

(a) (i) State the ratio test for the convergence of a real series with positive terms.

(ii) Define the radius of convergence of a real power series $\sum_{n=0}^{\infty} a_n x^n$.

(iii) Prove that the real power series $f(x) = \sum_n a_n x^n$ and $g(x) = \sum_n (n+1) a_{n+1} x^n$ have equal radii of convergence.

(iv) State the relationship between $f(x)$ and $g(x)$ within their interval of convergence.

(b) (i) Prove that the real series

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad g(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

have radius of convergence ∞.

(ii) Show that they are differentiable on the real line $\mathbb{R}$, with $f' = -g$ and $g' = f$, and deduce that $f(x)^2 + g(x)^2 = 1$.

[You may use, without proof, general theorems about differentiating within the interval of convergence, provided that you give a clear statement of any such theorem.]

END OF PAPER