Faculty of Mathematics
Part III Essays: 2016 - 2017

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Department of Pure Mathematics
& Mathematical Statistics

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& Theoretical Physics

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**Introductory Notes**

**General advice.** Before attempting any particular essay, candidates are advised to meet the setter in person. Normally candidates may consult the setter up to three times before the essay is submitted. The first meeting may take the form of a group meeting at which the setter describes the essay topic and answers general questions.

**Choice of topic.** The titles of essays appearing in this list have already been announced in the Reporter. If you wish to write an essay on a topic not covered in the list you should approach your Part III Adviser or any other member of staff to discuss a new title. You should then ask your Director of Studies to write to the Secretary of the Faculty Board, c/o the Undergraduate Office at the CMS (Room B1.28) **no later than 1 February**. The new essay title will require the approval of the Examiners. It is important that the essay should not substantially overlap with any course being given in Part III. Additional Essays will be announced in the Reporter no later than 1 March and are open to all candidates. Even if you request an essay you do not have to do it. Essay titles cannot be approved informally: the only allowed essay titles are those which appear in the final version of this document (on the Faculty web site).

**Originality.** The object of a typical essay is to give an exposition of a piece of mathematics which is scattered over several books or papers. Originality is not usually required, but often candidates will find novel approaches. All sources and references used should be carefully listed in a bibliography.

**Length of essay.** There is no prescribed length for the essay in the University Ordinances, but the general opinion seems to be that 6,000-10,000 words is a normal length. If you are in any doubt as to the length of your essay please consult your adviser or essay setter.

**Presentation.** Your essay should be legible and may be either hand written or produced on a word processor. Candidates are reminded that mathematical content is more important than style. Usually it is advisable for candidates to write an introduction outlining the contents of the essay. In some cases a conclusion might also be required. It is very important that you ensure that the pages of your essay are fastened together in an appropriate way, by stapling or binding them, for example.

**Credit.** The essay is the equivalent of one three-hour exam paper and marks are credited accordingly.

**Final decision on whether to submit an essay.** You are not asked to state which papers you have chosen for examination and which essay topic, if any, you have chosen until the beginning of the third term (Easter) when you will be sent the appropriate form to fill in and hand to your Director of Studies. Your Director of Studies should counter-sign the form and send it to the Chairman of Examiners (c/o the Undergraduate Office, Centre for Mathematical Sciences) so as to arrive before the second Friday in Easter Full Term, which this year is **Friday 5 May 2017**. **Note that this deadline will be strictly adhered to.**
Date of submission. You should submit your essay, through your Director of Studies, to the Chairman of Examiners (c/o Undergraduate Office, CMS). Your essay should be sent with the completed essay submission form found on page 10 of this document. The first part of the form should be signed by your Director of Studies. The second part should be completed and signed by you.

Please do not bind or staple the essay submission form to your essay, but instead attach it loosely, e.g. with a paperclip.

Then either you or your Director of Studies should take your essay and the signed essay submission form to the Undergraduate Office (B1.28) at the Centre for Mathematical Sciences so as to arrive not later than the second Friday in Easter Full Term, which this year is Friday 5 May 2017. Note that this deadline will be strictly adhered to.

Title page. The title page of your essay should bear ONLY the essay title. Please DO NOT include your name or any other personal details on the title page or anywhere else on your essay.

Signed declaration. The essay submission form requires you to sign the following declaration. It is important that you read and understand this before starting your essay.

I declare that this essay is work done as part of the Part III Examination. I have read and understood the Statement on Plagiarism for Part III and Graduate Courses issued by the Faculty of Mathematics, and have abided by it. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

Important note. The Statement on Plagiarism for Part III and Graduate Courses issued by the Faculty of Mathematics is reproduced starting on page 11 of this document. If you are in any doubt as to whether you will be able to sign the above declaration you should consult the member of staff involved in the essay. If they are unsure about your situation they should consult the Chairman of the Examiners as soon as possible. The examiners have the power to examine candidates viva voce (i.e. to give an oral examination) on their essays, although this procedure is not often used. However, you should be aware that the University takes a very serious view of any use of unfair means (plagiarism, cheating) in University examinations. The powers of the University Court of Discipline in such cases extend to depriving a student of membership of the University. Fortunately, incidents of this kind are very rare.

Return of essays. It is not possible to return essays. You are therefore advised to make your own copy before handing in your essay.

Further advice. It is important to control carefully the amount of time spent writing your essay since it should not interfere with your work on other courses. You might find it helpful to construct an essay-writing timetable with plenty of allowance for slippage and then try your hardest to keep to it.
Research. If you are interested in going on to do research you should, if possible, be available for consultation in the next few days after the results are published. If this is not convenient, or if you have any specific queries about PhD admissions, please contact the following addresses:

**Applied Mathematics & Theoretical Physics**

research@damtp.cam.ac.uk
DAMTP PhD Admissions,
Mathematics Graduate Office,
Centre for Mathematical Sciences,
Wilberforce Road,
Cambridge CB3 0WA,
United Kingdom.

**Pure Mathematics & Mathematical Statistics**

research@dpmms.cam.ac.uk
DPMMS PhD Admissions,
Mathematics Graduate Office,
Centre for Mathematical Sciences,
Wilberforce Road,
Cambridge CB3 0WB,
United Kingdom.
To the Chairman of Examiners for Part III Mathematics.

Dear Sir/Madam,

I enclose the Part III essay of .........................................................

Signed ......................................................... (Director of Studies)

I declare that this essay is work done as part of the Part III Examination. I have read and understood the Statement on Plagiarism for Part III and Graduate Courses issued by the Faculty of Mathematics, and have abided by it. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

Signed: ......................................................... Date: ..............................

Title of Essay: ......................................................... Essay Number: ..............................

Name: ......................................................... College: ..............................

Home address: (for return of comments) .........................................................

.........................................................
Appendix: Faculty of Mathematics Guidelines on Plagiarism

For the latest version of these guidelines please see

http://www.maths.cam.ac.uk/facultyboard/plagiarism/

University Resources

The University publishes information on Good academic practice and plagiarism, including

- a University-wide statement on plagiarism;
- Information for students, covering
  - Your responsibilities
  - Why does plagiarism matter?
  - Using commercial organisations and essay banks
  - How the University detects and disciplines plagiarism;
- information about Referencing and study skills;
- information on Resources and sources of support;
- the University’s statement on proofreading;
- FAQs.

There are references to the University statement

- in the Part IB and Part II Computational Project Manuals,
- in the Part III Essay booklet, and
- in the M.Phil. Computational Biology Course Guide.

Please read the University statement carefully; it is your responsibility to read and abide by this statement.

The Faculty Guidelines

The guidelines below are provided by the Faculty to help students interpret what the University Statement means for Mathematics. However neither the University Statement nor the Faculty Guidelines supersede the University’s Regulations as set out in the Statutes and Ordinances. If you are unsure as to the interpretation of the University Statement, or the Faculty Guidelines, or the Statutes and Ordinances, you should ask your Director of Studies or Course Director (as appropriate).

What is plagiarism?

Plagiarism can be defined as the unacknowledged use of the work of others as if this were your own original work. In the context of any University examination, this amounts to passing off the work of others as your own to gain unfair advantage.

Such use of unfair means will not be tolerated by the University or the Faculty. If detected, the penalty may be severe and may lead to failure to obtain your degree. This is in the interests of the vast majority of students who work hard for their degree through their own efforts, and it is essential in safeguarding the integrity of the degrees awarded by the University.
Checking for plagiarism

Faculty Examiners will routinely look out for any indication of plagiarised work. They reserve the right to make use of specialised detection software if appropriate (the University subscribes to Turnitin Plagiarism Detection Software). See also the Board of Examinations' statement on How the University detects and disciplines plagiarism.

The scope of plagiarism

Plagiarism may be due to

- **copying** (this is using another person’s language and/or ideas as if they are your own);
- **collusion** (this is collaboration either where it is forbidden, or where the extent of the collaboration exceeds that which has been expressly allowed).

How to avoid plagiarism

Your course work, essays and projects (for Parts IB, II and III, the M.Phil. etc.), are marked on the assumption that it is your own work: i.e. on the assumption that the words, diagrams, computer programs, ideas and arguments are your own. Plagiarism can occur if, without suitable acknowledgement and referencing, you take any of the above (i.e. words, diagrams, computer programs, ideas and arguments) from books or journals, obtain them from unpublished sources such as lecture notes and handouts, or download them from the web.

Plagiarism also occurs if you submit work that has been undertaken in whole or part by someone else on your behalf (such as employing a ‘ghost writing service’). Furthermore, you should not deliberately reproduce someone else’s work in a written examination. These would all be regarded as plagiarism by the Faculty and by the University.

In addition you should not submit any work that is substantially the same as work you have submitted, or are concurrently submitting, for any degree, diploma or similar qualification at any university or similar institution.

However, it is often the case that parts of your essays, projects and course-work will be based on what you have read and learned from other sources, and it is important that in your essay or project or course-work you show exactly where, and how, your work is indebted to these other sources. The golden rule is that the Examiners must be in no doubt as to which parts of your work are your own original work and which are the rightful property of someone else.

A good guideline to avoid plagiarism is not to repeat or reproduce other people’s words, diagrams or computer programs. If you need to describe other people’s ideas or arguments try to paraphrase them in your own words (and remember to include a reference). Only when it is absolutely necessary should you include direct quotes, and then these should be kept to a minimum. You should also remember that in an essay or project or course-work, it is not sufficient merely to repeat or paraphrase someone else’s view; you are expected at least to evaluate, critique and/or synthesise their position.

In slightly more detail, the following guidelines may be helpful in avoiding plagiarism.

**Quoting.** A quotation directly from a book or journal article is acceptable in certain circumstances, provided that it is referenced properly:
• short quotations should be in inverted commas, and a reference given to the source;
• longer pieces of quoted text should be in inverted commas and indented, and a reference given to the source.

Whatever system is followed, you should additionally list all the sources in the bibliography or reference section at the end of the piece of work, giving the full details of the sources, in a format that would enable another person to look them up easily. There are many different styles for bibliographies. Use one that is widely used in the relevant area (look at papers and books to see what referencing style is used).

**Paraphrasing.** Paraphrasing means putting someone else’s work into your own words. Paraphrasing is acceptable, provided that it is acknowledged. A rule of thumb for acceptable paraphrasing is that an acknowledgement should be made at least once in every paragraph. There are many ways in which such acknowledgements can be made (e.g. “Smith (2001) goes on to argue that ...” or “Smith (2001) provides further proof that ...”). As with quotation, the full details of the source should be given in the bibliography or reference list.

**General indebtedness.** When presenting the ideas, arguments and work of others, you must give an indication of the source of the material. You should err on the side of caution, especially if drawing ideas from one source. If the ordering of evidence and argument, or the organisation of material reflects a particular source, then this should be clearly stated (and the source referenced).

**Use of web sources.** You should use web sources as if you were using a book or journal article. The above rules for quoting (including ‘cutting and pasting’), paraphrasing and general indebtedness apply. Web sources must be referenced and included in the bibliography.

**Collaboration.** Unless it is expressly allowed, collaboration is collusion and counts as plagiarism. Moreover, as well as not copying the work of others you should not allow another person to copy your work.

**Links to University Information**

• Information on *Good academic practice and plagiarism*, including
  - *Information for students*.
  - information on *Policy, procedure and guidance for staff and examiners*.
Table 1: A Timetable of Relevant Events and Deadlines

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<td>Deadline for Candidates to request additional essays.</td>
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<tr>
<td>Friday 5 May</td>
<td>Deadline for Candidates to return form stating choice of papers and essays.</td>
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<tr>
<td>Friday 5 May</td>
<td>Deadline for Candidates to submit essays.</td>
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<tr>
<td>Thursday 1 June</td>
<td>Part III Examinations begin.</td>
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Comments. If you feel that these notes could be made more helpful please write to The Chairman of Examiners, c/o the Undergraduate Office, CMS.

Further information. Professor T.W. Körner (DPMMS) wrote an essay on Part III essays which may be useful (though it is slanted towards the pure side). It is available via his homepage

1. Combinatorial Algorithms in the Representation Theory of Symmetric Groups

Dr S. Martin

In my lecture course you will understand that many fundamental results about representations of symmetric groups [1] can be derived in a purely combinatorial manner. A good example of this phenomenon is the theorem asserting that the dimension, $f^\lambda$, of the Specht module (equivalently, the degree of the ordinary irreducible character corresponding to the cycle type $\lambda$) equals the number of standard Young tableaux of shape $\lambda$.

The essay could deal with treatments of other algorithmic proofs of well known results for symmetric groups, for example:

(a) algorithms to prove the Frame-Robinson-Thrall hook formula and the Frobenius-Young determinantal formula;

(b) generalising the hook-rule formula by considering a hook generating function for semistandard tableaux using a wonderful algorithm due to Hillman and Grasl;

(c) the Robinson-Schensted-Knuth algorithm, which provides a bijective proof that $\sum_{\lambda\vdash n} (f^\lambda)^2 = n!$;

(d) Schützenberger’s jeu de taquin (the ‘teasing game’) which gives an alternative approach to (c) and to the so-called Knuth relations.

There is a gentle account of this in Chapter 3 of Sagan’s book [3]; there are also some interesting illustrative but substantial exercises at the end of his chapter which could be incorporated into part of the body of the essay. A whole host of weird and wonderful gems also appears in Chapters 11 and 12 of [4].

Relevant Courses

Essential: Representation Theory

Useful: Lie algebras; Algebras; knowledge of ordinary character theory, e.g. from [2].

References


2. Symmetric Functions

Dr S. Martin

Results on representations of symmetric groups can be proved using very basic facts from representation theory (such as the fact that an irreducible module is always a composition factor of the group algebra), or we can deploy combinatorial methods in the spirit of the current Part
III course. There is however a third way using symmetric functions, very little of which has been presented in my course.

An important class of symmetric function is the Schur function, with the Jacobi-Trudi determinants for the Schur functions mirroring the determinantal formula for calculating the dimension $f^A$ of the Specht module. Other applications include the famous Littlewood-Richardson Rule for decomposing tensor products into irreducible summands and the Murnaghan-Nakayama Rule which gives an algorithmic way to calculate characters.

The best account of symmetric function theory is in Chapter I of Macdonald [1]; an alternative is Chapter 7 of the encyclopedic book of Stanley [2], which also includes a lot of applications and exercises.

Relevant Courses

Essential: Representation theory, commutative algebra
Useful: ordinary character theory and basic group theory

References


3. Complexes of Curves .................................................................

Dr H. Wilton

The mapping class group $\text{Mod}(\Sigma)$ of a compact surface $\Sigma$ plays a fundamental role in diverse areas of mathematics. The structure of $\text{Mod}(\Sigma)$ is typically extremely complicated, but one powerful technique for analysing it is provided by the complex of curves $C(\Sigma)$.

The goal of this essay is to prove some geometric and topological properties of $C(\Sigma)$, and to show how they can be applied to study $\text{Mod}(\Sigma)$.

Relevant Courses

Essential: Part II Algebraic topology, Part III Geometric group theory

References


4. Rigidity Theorems for Hyperbolic Groups .........................

Dr H. Wilton

A finitely generated group is called \textit{word-hyperbolic} if triangles in its Cayley graph are uniformly thin. This condition defines a vast class of groups, first introduced by Gromov [5], and enables the geometric techniques developed by Thurston when studying hyperbolic 3-manifolds to be applied in a much wider setting.

The idea of the essay is to explore rigidity theorems for hyperbolic groups. A typical result is Paulin’s theorem, which says that the outer automorphism group of a torsion-free hyperbolic group $\Gamma$ is infinite if and only if $\Gamma$ splits as an amalgamated free product or HNN extension over a cyclic subgroup [3,6].

A successful essay, after describing some of the basic theory of hyperbolic groups [4 Chapters III.H and III.I], will explain the basic strategy used to prove theorems like Paulin’s theorem (sometimes called the \textit{Bestvina-Paulin method}): if rigidity fails (in this case, if the outer automorphism group is infinite), then one can apply a limiting argument to extract an action on an $\mathbb{R}$-tree [1]; one then applies Rips’ classification of actions on $\mathbb{R}$-trees [2] to deduce a contradiction. More advanced essays will go into other results in the same vein, such as Rips-Sela’s proof that rigid hyperbolic groups are co-Hopfian [7].

Relevant Courses

\textit{Essential:} Part II Algebraic topology, Part III Geometric group theory

References

http://www.math.utah.edu/~bestvina/eprints/handbook.ps


5. Outer Space ......................................................... Dr H. Wilton

The action of a group \( \Gamma \) on itself by conjugation defines a natural map \( \Gamma \to \text{Aut}(\Gamma) \). Its image is the (normal) subgroup of inner automorphisms, and the quotient \( \text{Aut}(\Gamma)/\text{Inn}(\Gamma) \) is called the outer automorphism group \( \text{Out}(\Gamma) \). When \( \Gamma = F_n \), the non-abelian free group of rank \( n \), the group \( \text{Out}(F_n) \) is especially complicated and interesting.

An important idea in geometric group theory is that one can study interesting groups by constructing nice spaces on which they act. For \( \text{Out}(F_n) \), Culler and Vogtmann [4] constructed a certain space of graphs, now known as ‘Culler–Vogtmann Outer Space’ and denoted by \( \mathcal{CV}_n \). Topological properties of \( \mathcal{CV}_n \) translate into group-theoretic properties of \( \text{Out}(F_n) \). For instance, Culler and Vogtmann showed that \( \mathcal{CV}_n \) is contractible, from which it follows that \( \text{Out}(F_n) \) has finite cohomological dimension.

The idea of this essay is to describe the construction of \( \mathcal{CV}_n \), to give a proof that it is contractible, and to deduce the corresponding results about \( \text{Out}(F_n) \). The original Culler–Vogtmann proof of contractibility is quite combinatorial, but a more transparent geometric proof was given by Skora — see [3] or [5]. The required results about cohomological dimension can be found in [2]. An excellent essay might go on to describe more general deformation spaces (along the lines of [3] or [5], or to prove the existence of train tracks [1].

Relevant Courses

*Essential:* Part II Algebraic topology, Part III Geometric group theory

*Useful:* Part III Algebraic topology

References


6. Birational Geometry of Arithmetic Surfaces .......................... Professor C. Birkar

An arithmetic surface is a two-dimensional scheme of finite type over \( \text{Spec } \mathbb{Z} \). Such schemes play a fundamental role in arithmetic geometry and serve as a bridge between algebraic geometry and number theory. This essay is focused on the birational geometry of arithmetic surfaces and
applications to construction of Néron models of elliptic curves.

Let $R$ be a Dedekind domain and $K$ its fraction field (for example $R = \mathbb{Z}$ and $K = \mathbb{Q}$). Consider an elliptic curve $E$ over $K$. A Néron model for $E$ is essentially a “nice minimal” arithmetic surface $X$ over $R$ whose generic fibre is $E$.

The candidate is expected to discuss the following:

- Classical birational geometry of surfaces over algebraically closed fields [1, Chapter V]: This should include blowups of smooth points (monoidal transformations), Castelnuovo’s criterion [1, Chapter V, Theorem 5.7] with proof, and existence of minimal models [1, Chapter V, Theorem 5.8] with proof.

- Birational geometry of arithmetic surfaces [2] or [3] ([2] is recommended as it is more recent): This should include blowups [2, Chapter 8], Castelnuovo’s criterion [2, Chapter 9, Theorem 3.8] with proof, existence of minimal regular models [2, Chapter 9, Theorems 3.19 and 3.21] with proofs, plus necessary background material.

- Néron models of elliptic curves [4, Chapter IV]: Brief discussion of existence of proper regular models [4, Chapter IV, Theorem 4.5(a)], Néron models and some of their basic properties [4, Chapter IV, Section 5], sketch of proof of existence of Néron models [4, Chapter IV, Theorem 6.1], plus necessary background material (eg, group schemes, [4, Chapter IV]).

**Relevant Courses**

**Essential:** Part III algebraic geometry.

**References**


**7. The Modal Logic of Grounds ..........................................................**

Professor B. Löwe

If $M$ and $N$ are countable transitive models of set theory, we say that $M$ is a *ground* of $N$ if there is a $\mathbb{P} \in M$ and a $\mathbb{P}$-generic filter $G$ over $M$ such that $N = M[G]$. Hamkins and Löwe studied the *modal logic of grounds* in [2] and proved that it is contained in $S4.2$ using Reitz’s bottomless universe. The paper [2] builds on techniques of Hamkins, Leibman, Löwe, and Reitz [1,3,4] and provides proof sketches of the main construction. The authors of [2] were not able to completely determine the modal logic of grounds: they showed that it was equal to $S4.2$ if the so-called *downward directedness of grounds hypothesis* holds, but they were unable to prove that hypothesis. In early 2016, Usuba announced that he had proved the downward directedness of grounds hypothesis [5].
The goal of this essay will be to spell out the details of the proof sketch in [2] and combine it with Usuka’s result on the downward directedness of ground hypothesis to determine the modal logic of grounds.

**Relevant Courses**

*Essential*: Part II Logic & Set Theory (or equivalent), Part III Topics in Set Theory (or equivalent).

**References**


8. Games with Slightly Imperfect Information on Arbitrary Sets of Moves .

Professor B. Löwe

Blackwell introduced *infinite games with slightly imperfect information* in his paper [1]. In these games, two players play infinitely many one-round imperfect information games and such a game is called Blackwell determined if both players have *optimal mixed strategies* that can enforce the expected outcome with arbitrary precision. One can show that games with slightly imperfect information are in general not Blackwell determined if the set of possible moves is infinite.

Games with slightly imperfect information gave rise to a set-theoretic principle called *Blackwell Determinacy* (due to Martin and Vervoort). As opposed to Blackwell’s original games, the notion of Blackwell determinacy in this setting makes sense for infinite sets of moves [2].

In classical determinacy theory, if the set of moves is too big, determinacy principles become inconsistent. E.g., determinacy for games on the natural numbers or the real numbers are consistent with ZF assuming the consistency of large cardinals; yet, determinacy for games on \( \omega_1 \), the smallest uncountable ordinal, or determinacy for games on \( \varphi(\mathbb{R}) \), the power set of the real numbers, are inconsistent with ZF.

The task of this essay is to explore the same phenomenon for Blackwell Determinacy principles and prove the inconsistency of Blackwell determinacy for games with moves in \( \omega_1 \). This is an unpublished result by Ikegami based on preliminary work contained in the survey paper [3].
Relevant Courses

Essential: Part II Logic and Set Theory (or equivalent), Part III Topics in Set Theory (or equivalent), Part II Probability and Measure (or equivalent).

Useful: —.

References


9. Transcendence Results in the style of Solovay and Judah-Shelah 

Professor B. Löwe

Sets exhibiting lack of regularity (e.g., non-Lebesgue measurable sets or sets without the Baire property) are constructed using the Axiom of Choice.

In Gödel’s constructive universe $L$, there is a definable well-ordering of the real numbers which gives a definable way to use the Axiom of Choice and consequently get definable sets exhibiting lack of regularity. E.g., in $L$, we have a $\Delta^1_2$ set that is not Lebesgue measurable. Solovay, Judah, and Shelah showed that these constructions characterise the constructible universe in the following sense:

**Theorem** (Judah-Shelah; [1]). The statement “every $\Delta^1_2$ set is Lebesgue measurable” is true if and only if for every real number $x$, there is a real generic over $L[x]$ for Solovay’s random forcing.

Such a theorem is called a *transcendence result* since it gives an equivalence between a regularity statement and a statement expressing that the universe transcends the constructible universe in a certain sense.

This essay aims to give an overview of transcendence results for various regularity properties. Forcing partial orders and regularity properties are connected in a systematic way that opens a way to formulate results in a rather abstract setting (cf. the PhD dissertations of Khomskii and Iyegami [2,3]).

Relevant Courses

Essential: Part II Logic and Set Theory (or equivalent), Part III Topics in Set Theory (or equivalent).

Useful: Part Ib Metric and Topological Spaces (or equivalent), Part II Probability and Measure (or equivalent).
The theory of logarithmic schemes was developed in the 1980s by Illusie-Fontaine and Kazuya Kato. As described by Kato, a logarithmic structure on a scheme is a “magic powder” which makes relatively nice singular schemes look smooth. A typical example is a normal crossings divisor, which formally looks smooth if viewed as a log scheme.

While the original motivation for introducing log schemes was for its arithmetic applications, more recently log schemes have found powerful applications in mirror symmetry. This essay should cover the fundamentals of log geometry, and then explore applications of interest to the essay writer. A natural direction to explore is the theory of logarithmic Gromov-Witten invariants, which give a log version of curve counting. Without developing the full technology of Gromov-Witten invariants, the writer could explore Nishinou and Siebert’s proof of Mikhalkin’s tropical curve counting result [5].

The original papers [1], [2] are dense but readable. Chapter 3 of [3] contains a more relaxed introduction to parts of the theory needed for mirror symmetry. [4] is a partial encyclopedic manuscript on the subject. These sources are more than enough to get started. Chapter 4 of [3] contains a proof of the result of Nishinou-Siebert [5] in the two-dimensional case, and [6] develops the theory of logarithmic Gromov-Witten invariants; the most ambitious essay writer would take some of the latter material on board, but this will require some willingness to at least wave ones hands at algebraic stacks.

Relevant Courses

Essential: Part III Algebraic Geometry, Useful: Topics in Algebraic Geometry (Lent term)

References


11. Tropical Geometry ........................................... Professor M. Gross

Tropical geometry is algebraic geometry over the so-called tropical semiring, the semiring of real numbers with addition being maximum and multiplication being ordinary addition. Thus a “tropical polynomial” in \(n\) variables is really a function given as a maximum of a collection of affine linear functions with integral slopes. The “zero locus” of such a function is interpreted as the locus where such a function isn’t linear. For example, we define a tropical hypersurface in \(\mathbb{R}^n\) as the non-linear locus of such a function.

While these resulting objects are very combinatorial in nature, there turns out to be a rich and surprising relationship between tropical geometry and complex geometry. For example, Mikhalkin [1] really started the subject by showing that curves in the complex projective plane can be counted tropically.

There is now a wide literature in the subject, and some of this literature does not require very much background in algebraic geometry. Thus this essay should be accessible to students who have not taken the Part III Algebraic Geometry class, as long as they are willing to learn a little bit of algebraic geometry on the way.

Chapter 1 of [2] gives an introduction to tropical geometry, and Chapter 4 gives the most technologically advanced proof of Mikhalkin’s result. [3] gives a very elementary introduction to the subject, and [4] gives an elementary, completely combinatorial proof (but see Chapter 2 of [2] for the necessary background). Depending on the tastes of the essay writer, various aspects of the theory can be explored. Possibilities include (a) the relationship between tropical geometry and amoebas [5]; (b) enumerative applications [1], [4]; (c) applications to mirror symmetry, especially [2], Chapter 5.

Relevant Courses

Useful: Part III Algebraic Geometry (Michaelmas term), Topics in Algebraic Geometry (Lent term)

References

12. Picard-Lefschetz theory ........................................ Professor I. Smith

Picard-Lefschetz theory involves studying an algebraic variety, or symplectic manifold, via a pencil of hypersurface sections. This essay will start by discussing the algebraic topology of Lefschetz pencils, Dehn twist monodromy, and the existence of Lefschetz pencils on algebraic varieties. There are then several possible directions: constraints on the topology of Lefschetz pencils on four-manifolds; Milnor fibres of isolated hypersurface singularities; the geometry of varieties with small dual; Donaldson’s existence theorem for Lefschetz pencils on symplectic manifolds, etc, depending on the background and interests of the writer.

Relevant Courses

Essential: Algebraic topology, differential geometry.
Useful: Algebraic geometry, Linear systems, Stein manifolds and symplectic cohomology (graduate course).

References


13. Non-Squeezing ................................................... Professor I. Smith

Gromov’s non-squeezing theorem asserts that one cannot embed a fat symplectic ball into a thin symplectic cylinder. It can be interpreted as an uncertainty principle for classical mechanics, and is emblematic of a large array of “embedding obstructions” in symplectic topology which have no counterpart for volume-preserving maps. The theorem is most efficiently proved using the theory of pseudo-holomorphic curves. This essay will (modulo some analytic background) explain the basics of holomorphic curve theory on symplectic manifolds, deduce the non-squeezing theorem, and then describe some more recent “symplectic packing” results.

Relevant Courses

Essential: Algebraic topology, differential geometry.
Useful: Algebraic geometry, Stein manifolds and symplectic cohomology (graduate course).
References


14. Categories of Relations

Categories whose morphisms behave like relations rather than functions can be studied in various ways. The objective of the study is to identify those morphisms (commonly called maps) which correspond to actual functions, and to relate properties of the whole category to properties of its subcategory of maps. One highly successful approach, originally developed by Peter Freyd, is developed in detail in [1], and more succinctly in [2]; other approaches include that of Carboni and Walters [3], which makes more explicit use of 2-categorical ideas. It is suggested that an essay might take as its goal the characterization of those allegories (or cartesian bicategories) whose categories of maps are toposes; alternatively, one might give a detailed comparison of these two (and possibly other) approaches.

Relevant Courses

Essential: Category Theory

References


15. Locally Presentable and Accessible Categories

Locally presentable categories were introduced by Gabriel and Ulmer [1,2], and were an early attempt to capture the essential categorical structure of the category of models of a theory. The fact that they succeeded in doing just this, for a particular (very natural) class of ‘essentially algebraic’ theories, was proved by M. Coste [3]. More recently, attention has focused on the much larger class of accessible categories [4,5], which are categories of models of theories in a much broader sense; locally presentable categories are precisely those accessible categories which are complete as categories. An essay on this topic could either take as its goal the main theorem characterizing accessible categories as categories of models, or it could survey the way in which particular properties of the axiomatization of a theory are reflected in properties of its category of models.

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Relevant Courses

**Essential:** Category Theory

References


16. Algebraic $K$-theory .........................................................

Dr O. Randal-Williams

The algebraic $K$-groups $K_n(R)$ of a ring $R$ are a sort of “homology theory” for rings. The group $K_0(R)$ controls the stable classification of projective $R$-modules; the group $K_1(R)$ is a universal target for determinant-like invariants of automorphisms of projective $R$-modules. The higher $K$-groups cannot be described so simply: they were defined by Quillen using—perhaps surprisingly—techniques from algebraic topology. Quillen constructed a topological space $K(R)$, and defined $K_n(R) := \pi_n(K(R))$ as the higher homotopy groups of this space. In fact, he gave several definitions of spaces $K(R)$ and showed them to be homotopy equivalent: these multiple points of view make the study of algebraic $K$-theory a very rich subject, with connections to algebraic geometry, arithmetic, and the topology of manifolds.

You should start by briefly surveying the definitions and basic properties of the lower $K$-groups. You should then introduce the necessary categorical background and explain Quillen’s “$Q$”-construction of $K(R)$ in [1], and then discuss its basic properties (additivity, resolution, dévissage, localisation). You should then explain Quillen’s “$+$”-construction of $K(R)$, and explain in detail why it gives a homotopy equivalent space, by comparing both with the “$S^{-1}S$”-construction.

Finally, you could briefly survey Quillen’s calculation of the higher $K$-theory of finite fields, and Borel’s calculation of the higher $K$-theory of rings of integers in number fields.

Relevant Courses

**Essential:** Algebraic Topology

**Useful:** Category theory, Algebras
References


17. Exotic Spheres ................................................................. Dr O. Randal-Williams

In 1956 Milnor constructed certain 7-dimensional smooth manifolds which are homeomorphic to $S^7$ but not diffeomorphic to it: an “exotic sphere”. This was considered to be quite surprising. Later Kervaire and Milnor made a systematic study of the possible exotic spheres which can exist in any given dimension, showing that the classification of exotic spheres is closely related to a fundamental problem in homotopy theory: computing (stable) homotopy groups of spheres.

You should begin by explaining Milnor’s construction of exotic 7-spheres, following [1]. In order to do so you will have to understand a lot about characteristic classes, cobordism, and Hirzebruch’s Signature Theorem, for which you should consult [2].

There are then several directions you could pursue. You could use these and related ideas to construct a topological 8-manifold which does not admit any smooth structure. You could explain Brieskorn’s construction of exotic spheres from the link of certain singularities. For (many) more ideas see [3].

Relevant Courses

*Essential*: Algebraic Topology, Differential Geometry

References


18. The Classical Black Hole Stability Problem in General Relativity ...... Professor M. Dafermos

Black holes are one of the most spectacular predictions of general relativity. The concept is exhibited by the celebrated closed-form Schwarzschild and Kerr metrics. *But are these spacetimes stable as solutions to the Einstein equations?*

Much work has been done over the years in the direction of proving that they are. A complete resolution of this problem, however, even in the simpler linear setting, remains elusive.

The goal of this essay is to survey aspects of the classical black hole stability problem for the Schwarzschild and Kerr metrics. A good place to start is the proofs of “mode stability” by Whiting [9] and Shlapentokh-Rothman [8]. The essay can then either focus on the linear wave equation on Kerr (cf. [1,5]) or the equations of gravitational perturbations on Schwarzschild
(cf. [3]) but should certainly include a precise statement of how the full non-linear stability problem is formulated for Kerr.

**Relevant Courses**

*Essential:*
- General relativity, Black holes

*Useful:*
- Differential geometry, PDE’s, Cosmology

**References**


   Dr T. L. Kelly

The goal of this essay is to give a exposition on a systematic and rigorous proof that Hodge diamonds flipped for Batyrev-Borisov mirrors. Before this work, it was only shown in very limited cases that, for a n-dimensional Calabi-Yau variety $\mathcal{M}$ and its mirror $\mathcal{W}$ their Hodge numbers were flipped, i.e.,

\[ \dim H^p(\mathcal{M}, \Omega^q) = \dim H^{n-p}(\mathcal{W}, \Omega^q). \]

Batyrev and Borisov formulated a recipe using polytopes in toric varieties to make a huge number of mirror pairs and prove a general theorem.

Given a Calabi-Yau complete intersection in a Fano toric variety, one can find its mirror by doing a combinatorial operation on lattice polytopes to find another Calabi-Yau complete intersection.
in another toric variety. Batyrev and Borisov introduced a combinatorial tool called the $E$-function to introduce so called ‘stringy Hodge numbers’ which gave a toric way to compute cohomology for regular Calabi-Yau complete intersections in toric varieties.

This essay is to give a good exposition about the mirror construction and a clear explanation of the proof that Batyrev and Borisov gave. It is a great way to develop understanding of toric varieties, tools in combinatorial algebraic geometry, and a combinatorial viewpoint of mirror symmetry.

**Relevant Courses**

*Essential: Algebraic Geometry (Michaelmas)*

**References**


20. The Erdős Distinct-distances Problem

*Professor W. T. Gowers*

The Erdős distinct-distances problem asks how many distinct distances there must be among $n$ points in the plane. If the points are equally spaced along a line, then there are $n-1$ distinct distances, and if they are arranged in a $\sqrt{n} \times \sqrt{n}$ grid then this is reduced by a logarithmic factor. Erdős conjectured that one could not do much better: that is, that there will always be at least $n^{1-o(1)}$ distinct distances. He also made a stronger conjecture, the unit-distance problem, which asserts that no single distance can occur more than $n^{1+o(1)}$ times. Both conjectures were wide open until 2010, when Larry Guth and Nets Katz, in a remarkable breakthrough, proved the first of them using the polynomial method in a very surprising and novel way. The main purpose of this essay is to give an exposition of their proof, though ideally it will contain some discussion of related results as well. The unit-distances problem is still wide open: the best known upper bound is around $n^{4/3}$.

**Relevant Courses**

The use of polynomials will be discussed in this term’s Part III course in additive combinatorics, but the examples of its use will be very different and it is not necessary to have attended this course.
21. A-infinity Algebras .........................................................

Dr C. J. B. Brookes

A-infinity algebras (or strongly homotopy associative algebras) were invented in the 1960s by the topologist Stasheff for use in homotopy theory. By the 1990s they had appeared in representation theory, geometry and mathematical physics.

I suggest any potential essayist looks at the introductory articles by Keller [2, 3, 4], and aims to present the basic theory and consider an application or two, eg the paper of Berest and Chalykh [1].

Relevant Courses

Useful: Algebras

References


22. Cliques of Primes in Noetherian Algebras .................................

Dr C. J. B. Brookes

During the sixties and beyond, many ring theorists attempted to extend the structure theory of indecomposable injective modules which had been developed by Matlis for the commutative case. In the seventies Jategaonkar highlighted the connections between bimodules, localization at prime ideals and the structure of injective modules, and this prompted a lot of work concentrating on localization. His book [1] expounds much of this.

One way of encoding information is to define a directed graph, with vertices corresponding to the prime ideals of the Noetherian algebra and edges linking vertices if a certain type of extension of modules exists involving the corresponding primes. The primes associated to a component of this graph is called a clique.

As an introduction I suggest you read the essay by K.A.Brown in [3]. The basics of the theory of non-commutative Noetherian rings will be covered in my Algebras course but you may have to consult [2] for additional help.

It will probably be sensible to concentrate on the representation theory of some good class of algebras, for example the enveloping algebras of finite dimensional Lie algebras.
Relevant Courses

Essential: Algebras

Useful: Lie Algebras and their representations

References


23. Complex Manifolds and Hörmander Estimates

Dr J. Ross

This topic is to serve as an introduction to the topic of complex manifolds, with the aim of discussing the Hörmander technique for the $\overline{\partial}$-equation on such manifolds.

The essay should include a brief account of some of the standard material on complex manifolds, which builds on the theory of differentiable manifolds but with smooth functions replaced by holomorphic ones. On such spaces one has the notion of a holomorphic vector bundle, and the holomorphic line bundles play a particularly important role through their connection between the theory of complex manifolds and algebraic geometry.

Having done this, the essay should aim to discuss aspects of the $\overline{\partial}$-equation, first for open subsets of $\mathbb{C}$ (or $\mathbb{C}^n$) and then for sections of holomorphic line bundles on complex manifolds. In particular the fundamental Hörmander estimate should be discussed. There are many applications of this estimate (such as the Kodaira embedding theorem that links with projective algebraic geometry, or vanishing theorems of cohomology), and the essay should end by focusing on one or two of these.

Prerequisites

Essential: Part III Differential Geometry (essential)

Essential: A small amount of functional analysis (e.g. the idea of the space of $L^2$-integrable functions)

Possibly relevant: Part III Algebraic Geometry (only needed if one chooses to focus on algebraic aspects)

References

For the fundamentals of complex manifolds the following reference covers more than is needed:
For the Hörmander estimate and applications there are various sources that will likely be useful, depending on the chosen focus of the essay, such as

[2] Bo Berndtsson *An introduction to things* $\overline{\partial}$  
Available online [www.math.chalmers.se/~bob/7nynot.pdf](http://www.math.chalmers.se/~bob/7nynot.pdf)


24. Primitive Ideals in Enveloping Algebras ........................................  
Dr S. J. Wadsley  

Classifying all irreducible (not necessarily finite-dimensional) representations of a finite dimensional complex Lie algebra is a hopeless task. However, Dixmier [1] proposed studying a coarser classification — that of annihilators of irreducible representations, known as primitive ideals. This turns out to be both a reasonable and important task.

In this essay, you will focus mostly on the case is semisimple explaining the statements and proofs of the classification. Duflo was the first to prove in [2] that in this case every primitive ideal arises as the annihilator of an irreducible highest weight module. Joseph produced the first purely algebraic proof of this fact in [3]. See [4] for a survey by Joseph.

**Relevant Courses**

*Essential:* Lie Algebras and their representations, Algebras.

**References**


25. Auslander–Gorenstein Rings ........................................  
Dr S. J. Wadsley  

Lots of interesting rings are Auslander–Gorenstein. A ring is Auslander–Gorenstein if certain homological properties of its category of modules are well-behaved in specific ways.
I imagine an essay in this area would begin by discussing the necessary properties for a ring to be Auslander–Gorenstein and some of the immediate applications of them such as the definition of a canonical dimension function. You might then continue with a discussion of some of the natural examples probably including an explanation of why a Zariski-filtered ring with Auslander–Gorenstein graded ring is itself Auslander–Gorenstein — a result that yields lots of examples. You would then probably finish by considering how the Auslander–Gorenstein condition has been used to better understand some of these specific examples.

Relevant Courses

*Essential:* Algebras

References


26. Yau’s Solution of the Calabi Conjecture ................................................
Dr A. G. Kovalev

The subject area of this essay is compact Kähler manifolds. Very informally, a Kähler manifold is a complex manifold admitting a metric and a symplectic form, both nicely compatible with the complex structure. The Ricci curvature of a Kähler manifold may be equivalently expressed as a differential form which is necessarily closed. Furthermore, the cohomology class defined by this form depends only of the complex manifold, but not on the choice of Kähler metric. The Calabi conjecture determines which differential forms on a compact complex manifold can be realized by Ricci forms of some Kähler metric. Substantial progress on the conjecture was made by Aubin and it was eventually proved by Yau. This result gives, among other things, a powerful way to find many examples of Ricci-flat manifolds. The essay could discuss aspects of the proof and possibly consider some applications and examples. Interested candidates are welcome to contact A.G.Kovalev@dpmms for further details.

Relevant Courses

*Essential:* Differential geometry

*Useful:* Algebraic topology, Elliptic Partial Differential Equations
References


27. Atiyah–Singer Index Theorem

Dr A. G. Kovalev

The main object of study in this essay is elliptic differential operators. One well-known example of elliptic operator is the Laplacian, another, and perhaps more important in this essay, is a Dirac operator. The elliptic property can be defined for operators acting on functions, and more generally sections of vector bundles, over smooth manifolds. If a manifold is compact then every elliptic operator over it has finite-dimensional kernel and cokernel. The difference between these two latter dimensions is called an index. A celebrated theorem due to Atiyah and Singer asserts that the index of elliptic operator, an analytic quantity, can be computed entirely from topological invariants of the base manifold and vector bundle. Several different proofs of this theorem are known by now and the essay can discuss aspects of some proof and/or applications. Interested candidates are welcome to further discuss the possibilities with me (A.G.Kovalev@dpmms); the presentation [1] provides a nice introduction to the topic.

Relevant Courses

*Essential*: Algebraic topology, Differential geometry

*Useful*: Analysis of Partial Differential Equations, Elliptic partial differential equations

References


28. Rational Knots and the Slice-Ribbon Conjecture

Dr J. Rasmussen

A knot in $S^3$ is *smoothly slice* if it bounds a smoothly embedded disk in the four-ball. An old conjecture of Fox states that any such knot is *ribbon*: that is, it bounds an immersed disk in $S^3$ which can be pushed into $B^4$ to obtain an embedded disk. There are certain obstructions to a
knot being slice coming from algebraic topology, and in the 60’s Levine showed that these are the only obstructions to sliceness in high dimensions. Subsequently Casson and Gordon defined new obstructions which are specific to the case of knots in \( S^3 \); these invariants were one of the first examples of the special nature of topology in dimension 4.

Casson and Gordon computed their invariants for a simple family of knots known as rational knots. Based on their computations, they found some families of rational knots which were easily seen to be ribbon, and conjectured that these were the only ones. In the smooth case, this question was resolved in the affirmative by Lisca about 10 years ago, thus proving the slice-ribbon conjecture in the special case of rational knots. In the topological case, it remains open.

The aim of the essay is to learn something about this circle of ideas. The essay should first review some of the classical background about slice knots, including the notion of a knot being “algebraically slice,” and the relation between slice knots and rational homology spheres bounding rational homology balls. It should then focus on explaining the ideas behind one of a) Casson and Gordon’s original paper or b) Lisca’s paper (but not both). Depending on time and interest, you may also wish to consider the relation between the Casson-Gordon invariants of lens spaces and the \( d \)-invariants used by Lisca and/or the connection between slice rational knots and knots in the solid torus with solid torus surgeries.

\textbf{Relevant Courses}

\textit{Essential:} Algebraic Topology

\textit{Useful:} Differential Geometry

\textbf{References}


\textbf{29. The Coates–Wiles Theorem }.................................

\textit{Dr J. A. Thorne}

Let \( K \) be a number field, and let \( E \) be an elliptic curve over \( K \). The \( L \)-function \( L(E, s) \) is a holomorphic function of the complex variable \( s \in \mathbb{C} \), a priori defined in a right half-plane \( \text{Re} \ s > 3/2 \), but which conjecturally admits an analytic continuation to the whole complex plane, and satisfies a functional equation there (in much the same way as the Riemann zeta function \( \zeta(s) \)).

One of the outstanding conjectures of number theory is the Birch–Swinnerton-Dyer conjecture, which relates the order of vanishing of \( L(E, s) \) at the point \( s = 1 \) to the rank of the group \( E(K) \) (which is a finitely generated abelian group, by the Mordell–Weil theorem), and the leading coefficient in the Taylor expansion of \( L(E, s) \) at this point to finer arithmetic invariants of \( E \), such as the order of the Tate–Shafarevich group.
An elliptic curve is said to have complex multiplication if its endomorphism ring \( \text{End}(E) \) is strictly larger than \( \mathbb{Z} \). For these curves the analytic continuation of \( L(E, s) \) is known unconditionally. It is in this case that the first unconditional case of the Birch–Swinnerton-Dyer conjecture was established: Coates and Wiles showed that for an elliptic curve \( E \) over \( \mathbb{Q} \) with complex multiplication, the non-vanishing of \( L(E, 1) \) implies the finiteness of \( E(\mathbb{Q}) \).

The goal of this essay will be to explain Rubin’s proof of the Coates-Wiles theorem using the Euler system of elliptic units. If time allows, you could go on to describe his proof of the ‘Main Conjecture’ of Iwasawa theory in this setting using the same methods.

Relevant Courses

**Essential:** Elliptic Curves, Local Fields

**Useful:** Modular Forms

References


30. The Field of Norms .................................................................

Professor A. J. Scholl

Fontaine and Wintenberger discovered a remarkable connection between local fields of characteristic zero and those of characteristic \( p \). They showed that the absolute Galois group of a “sufficiently ramified” infinite extension of \( \mathbb{Q}_p \) is canonically isomorphic to the absolute Galois group of a local field of characteristic \( p \). This leads to a description of \( p \)-adic representations of the absolute Galois group of a \( p \)-adic local field in terms of so-called \((\phi, \Gamma)\)-modules, and an explicit description of their Galois cohomology. The aim of this essay will be to describe the construction of the “field of norms” and of \((\phi, \Gamma)\)-modules. Time permitting, the essay might also touch on one of the following:

- Herr’s proof [3] of Tate local duality using \((\phi, \Gamma)\)-modules;
- perfectoid fields [4].

Relevant Courses

**Essential:** Local fields.
The aim of this essay is to understand the statements, some of the proofs, and some of the consequences of the Hodge theorem and also its non-abelian generalisations.

Hodge theory is about the additional structures and constraints on the topology of an algebraic variety that arise from it being defined by equations. At its simplest, this is an additional structure on the cohomology of a smooth projective variety. One example of this is for an elliptic curve $E$, where the first cohomology group $H^1(E, \mathbb{C}) = \mathbb{C}^2$ has a canonical line inside it, $\Gamma(E, \Omega^1)$. As you vary the elliptic curve, the line moves, and in fact the position of this line is enough to recover the $j$-invariant of $E$ (and hence $E$).

Hodge theory is the best ‘linearisation’ of algebraic varieties that we currently understand.

You should begin with the abelian Hodge theorem and Lefschetz decomposition—understand the statement, some of its consequences, and one of its proofs (either the classical proof, which you can read in Griffiths and Harris, or by reduction mod p, due to Deligne-Illusie).

Varieties appear in families, so their Hodge structures do also. You may wish to read the classical theorems on how Hodge structures vary in families — Griffiths’ transversality, the regularity of singular points, quasi-unipotence of monodromy.

You can then go on and read the statements of non-abelian Hodge theory, how they generalise the abelian case, and perhaps some of its consequences.

Non-abelian Hodge theory has various technical underpinnings, some of which you should read and thoroughly understand, and some of which you should just take as given.

An ambitious goal might be to understand the paper of Reznikov on secondary invariants for vector bundles with flat connections.

References

The abelian Hodge theorem and its consequences are nicely explained in the textbook of Griffiths and Harris, or the textbook of Voisin. You should also look at the breathtakingly beautiful papers of Deligne.

For the non-abelian Hodge theorem, some references are:


And, as always, if you are thinking of doing this essay you should come and talk to me about it!

32. KZ Equations, Quantum Groups, Gal(\overline{Q}/Q), Periods .................

Professor I. Grojnowski

In the mid 1980s Drinfeld and Jimbo defined quantum groups, algebras $U_h$ over $C[[h]]$ which are deformations of the enveloping algebra $U_0$ of a semisimple Lie algebra.  

More precisely, $U_h$ is a Hopf algebra, free over $C[[h]]$, such that $U_h/hU_h$ is the enveloping algebra. Now, it is easy to see (and you will, in this essay!), that enveloping algebras of semisimple Lie algebras cannot deform — $U_h$ is isomorphic to $U_0 \otimes C[[h]]$, so one can think of this as saying what is actually changing is how you make the tensor product of two $U_h$-modules a $U_h$-module. One way of doing this, invented by Drinfeld, is to change the associativity constraint, that is the isomorphism $V_1 \otimes (V_2 \otimes V_3) \cong (V_1 \otimes V_2) \otimes V_3$ between the tensor product of three modules. And one way of doing this is to use the monodromy of the Knihzik-Zamalochikov equation, an explicit flat connection on vector bundles over $P^1 \setminus \{0,1,\infty\}$.  

The first main goal of this essay is to understand this precisely — that is, to understand Drinfeld’s theorem computing the deformations of $U_0$ as a bialgebra, the construction of the KZ associator, and the Drinfeld-Kohno theorem. This can all be found in the original papers, or in the lovely textbook by Etingof-Schiffmann.

You may then continue in two ways. You can learn more about the KZ-equation, the connection with unipotent local systems, special values of multi-zeta functions, and the Galois group of $\overline{Q}/Q$.

Or you may prefer to learn about more recent approaches to deformation theory — Kontsevich’s theorem, and its various proofs, and then the topologist’s heirarchy of homotopy commutative algebras, the $E_n$-algebras.

References

33. Derived Category, Stability Conditions, K3 Surfaces.  

Professor I. Grojnowski

The aim of this essay is to give an introduction to derived categories and homological algebra, as it is applied in algebraic geometry (homological mirror symmetry), and/or in representation theory.

As well as learning some abstract machinery, you will be gaining proficiency in either computing cohomology of coherent sheaves (geometry), or in representation theory (quivers, symplectic reflection algebras, finite groups).

There are various ways into this essay, for those with and without algebraic geometry.

If you know some algebraic geometry, then an ambitious aim for this essay would be to understand the papers of Bayer and Macri computing the nef and movable cones for moduli spaces of sheaves on a K3 surface.

Begin with Fourier-Mukai transform, which is an involution on the derived category of an elliptic curve. Use this to describe all vector bundles on an elliptic curve. More generally, do this for abelian varieties. A nice consequence is the Torelli theorem.

Then back up, and learn classical Koszul duality, the derived equivalence between modules for $SV$ and $^\Lambda V$; as well as Beilinson’s theorem describing all sheaves on $\mathbb{P}^n$.

You may then learn the theorem that every quasi-compact quasi-separated scheme is ‘derived affine’ (Bondal and van den Bergh, after Thomason and Neeman).

After this, try some explicit examples of flops by blowing up and down; this is due to Bondal-Orlov. You can continue in this way, and learn some of the derived category approach to birational geometry. Some lovely classical computations can be studied in this language; see for example the papers of Kuznetsov.

This is already enough for an essay, but if you’re a glutton for excess:

Now read Bridgeland’s papers on stability conditions. He constructs a complex variety attached to an abelian or derived category, each point of which parameterizes a t-structure and a notion of stability condition. You should compute these spaces for a line bundle over $\mathbb{P}^2$, at least.

You are now in a position to read the papers of Bridgeland, and Bayer and Macri, which compute this for a K3 surface, and relate it to the minimal model program.

If you don’t know any algebraic geometry, this is still a good essay—derived equivalences are at the heart of representation theory—but the examples you should work with are representations of groups, or quivers, or Hecke algebras. One very rich source of examples are the ‘cluster algebras’ of Fomin-Zelevinsky; Keller’s survey paper (listed below) is a fun introduction.

If you are interested in this essay, come talk to me and we’ll find a way into the subject that works best with your background.
34. Probabilistic Numerics

Probabilistic Numerics is an emerging field which deals with quantifying uncertainty in numerical algorithms for integration, optimisation, and the solution of differential equations. This project will be focused on Bayesian quadrature, an approach to estimating integrals $\Phi = \int f(x)\mu(dx)$, which might be difficult to compute because of the dimensionality of the space or the cost of evaluating the function $f$. A Gaussian Process prior is put on $f$, and the posterior distribution of $\Phi$ is used to evaluate the uncertainty in the numerical estimate. The theory of Bayesian quadrature is focused on proving how prior information about the function $f$ can lead to rates of convergence for the Bayes estimate superior to the usual $O(n^{-1/2})$ rate for Monte Carlo integration, and guarantees that the posterior contracts at the same rate.

The essay should review a number of recent papers by Oates, Briol, Girolami et al. [1-4], providing a concise account of their results. The review may be supplemented by theoretical developments, for example, working out rates of convergence in a novel reproducing kernel Hilbert space, or by a detailed analysis of a specific integration problem through mathematical analysis and simulation.

References


Relevant Courses

Essential: Part II Probability and Measure.

References
35. Deep Reinforcement Learning

Deep Reinforcement Learning is an approach to finding optimal policies in Markov Decision Processes. This technique has gained notoriety in Artificial Intelligence after several striking applications like AlphaGo, a program which defeated the world’s highest-ranked Go player in a 5 game championship, and a control system which reduced power consumption by 40% in Google’s server farms. Deep Reinforcement Learning is useful in areas like robotics where it is only possible to learn a policy through experience or simulation, either because the process is not known or because the state space is intractable. The technique involves using deep neural networks to model the value function and/or policy, and there are various algorithms for training based on stochastic gradient descent and Monte Carlo.

The goal of this essay is to produce a expository document on Deep Reinforcement Learning for an audience of mathematicians, which concisely explains the advantages and disadvantages of the most popular algorithms. The exposition can be complemented with simulations of a minimal example or toy problem which demonstrates the power of the method.

Relevant Courses

Essential: Part II Optimisation and Control
Useful: Part III Bayesian Modelling and Computation.

References


36. The Metric Theory of Type and Cotype

In 1976 Ribe proved a remarkable rigidity theorem for Banach spaces: if two Banach spaces are uniformly homeomorphic then every finite-dimensional subspace of one space is linearly isomorphic to a subspace of the other space with isomorphism constant independent of dimension. Consequently, local properties of a Banach space, i.e., properties dependent only on the finite-dimensional subspaces of the space, are determined by the metric structure of the space alone.
and can be formulated without reference to the linear structure. In 1986 Bourgain proposed what became known as “The Ribe Programme”: to find explicit metric descriptions of important invariants of Banach spaces. This led to non-linear versions of several concepts and results from Banach space theory which then had far-reaching consequences in other areas, notably in computer science and in geometric group theory.

One of the most successful achievements in this programme has been the development of the metric version of the theory of type and cotype. Metric type was invented by Enflo [3] even before the appearance of the linear theory. His definition was later modified by Bourgain, Milman and Wolfson: in [2] they proved a non-linear version of Pisier’s $\ell_1$ theorem on the obstruction to a space having non-trivial type. The next phase of the programme was the introduction of Markov type and cotype by Ball to deal with the extension of Lipschitz maps. The linear version is the famous extension theorem of Maurey which can be viewed as a generalization of Kwapień’s theorem characterizing Hilbert space in terms of linear type and cotype. Johnson and Lindenstrauss asked if an analogue of Maurey’s theorem holds for Lipschitz maps from subsets of $L_q$ into $L_p$ ($1 \leq p \leq 2 \leq q < \infty$) in effect asking for a metric version of type and cotype. The answer was Ball’s Markov type and cotype, but it took another 10 years before Naor, Peres, Schramm and Sheffield [5] completed the solution by establishing that $L_q$ has Markov type 2.

Although Ball’s definition does provide a metric notion of cotype which works for problems of extension of Lipschitz maps, it does not quite fit the requirements of the Ribe Programme. Metric cotype which generalizes linear cotype was finally found by Mendel and Naor [4]. The test for this notion is the Maurey-Pisier theorem that identifies the obstruction to a space having non-trivial cotype. Mendel and Naor proved the non-linear version of this theorem.

The purpose of this essay is to follow the development of metric type and cotype and to present some of the applications as outlined above. The articles of Ball [1] and Naor [6] are very useful background reading and provide an overview of the Ribe Programme. The main references for the essay itself are [2], [3], [4] and [5].

Relevant Courses

**Essential:** Part II Linear Analysis and Part III Functional Analysis (but only a very small part of these courses is needed).

**Useful:** IA Probability and IB Markov Chains.

References


37. Continuum Random Trees ........................................... Dr J. Miller

The continuum random tree (CRT) [1], introduced by Aldous, is a universal model for a continuum planar tree. Just like the Brownian motion arises as the scaling limit of many different discrete, random curves, the CRT arises as the scaling limit of many different types of discrete, random trees.

There are several constructions of the CRT. The starting point of the original construction due to Aldous is the so-called Brownian excursion $B$ defined on $[0,1]$. The CRT results by considering the graph $G = \{(t,B_t) : t \in [0,1]\}$ of $B$ and then identifying the points of $G$ which can be connected by a horizontal chord which lies entirely below $G$. In another construction, called “stick-breaking” and also due to Aldous, one divides the positive reals into a collection of segments (“sticks”) of random length and then glues them together to produce a random tree.

The CRT is also closely connected with several other universal objects in two-dimensional probability. For example, it serves as a building block in the construction of the so-called Brownian map [2,3], the continuum limit of random quadrangulations. It turns out that one can produce interesting structures by gluing two CRTs together. Surprisingly, the resulting object has the topology of a sphere and the interface between the two trees is one of Schramm’s SLE curves [4].

A successful essay will give the background on discrete and continuum random trees, the Brownian excursion, different constructions of the CRT, the proof that uniformly random trees converge to the CRT, as well as some discussion of random structures which can be built from the CRT.

Relevant Courses

Advanced probability.

References


38. Erdös-Rényi Random Graphs ........................................... Dr S. Andres

The study of complex networks plays an increasingly important role in science, where it has turned out that probability theory can offer an effective way to deal with such networks through the study of random graphs.
One of the simplest imaginable random graphs is the Erdős-Rényi graph \( G(n, p) \), which is obtained from the complete graph on \( n \) vertices by retaining each edge with probability \( p \) and deleting it with probability \( 1 - p \), independently of all other edges.

The behaviour of the largest connected component in \( G(n, p) \) have been of particular interest as it exhibits a phase transition in \( p \). More precisely, for \( p < \frac{1}{n} \) the size of the largest connected component is of logarithmic order, for \( p > \frac{1}{n} \) it is of order \( n \) and there is a critical window around the value \( p = \frac{1}{n} \), where it is approximately \( n^{2/3} \).

A successful essay will review these results including some detailed proofs, based on a coupling between the Erdős-Rényi graph and branching processes. One focus should be on the proof of the results in [1].

**Relevant Courses**

*Essential:* Probability and Measure, Markov Chains.

*Useful:* Graph Theory, Percolation and Random Walks on Graphs, Mixing Times of Markov Chains.

**References**


39. **The Grunwald-Wang Theorem**

Dr T. A. Fisher

A natural local-to-global question in number theory is whether an element of a number field \( k \) that is locally (i.e. in the completion \( k_v \)) an \( n \)th power for all but finitely many places \( v \), must be an \( n \)th power in \( k \). In general the answer is “no” (there are counter-examples with \( k = \mathbb{Q} \) and \( n = 8 \)) but under some extra hypotheses (which are always satisfied for example if \( n \) is odd) the answer is “yes”. The Grunwald-Wang theorem itself is a closely related local-to-global question about the existence of cyclic extensions of \( k \) (i.e. Galois extensions with cyclic Galois group). The essay should include sufficient background material from class field theory and group cohomology to explain the proofs. It could also discuss how the above problem (concerning divisibility in the multiplicative group) extends to other commutative algebraic groups [3], for example elliptic curves.

**Relevant Courses**

*Essential:* Local Fields

*Useful:* Elliptic Curves
References


40. High-dimensional Online Changepoint Detection

Professor R. J. Samworth and Dr T. Wang

Changepoint detection is a classical statistical problem, dating back at least to [1] in the univariate case; see also [2], [3], for example. However, modern applications in internet traffic monitoring, fMRI and finance, to name just a few, have motivated a resurgence of interest in the topic, from a high-dimensional perspective. One interesting model (e.g. [4]) is where changes occur only in a sparse subset of coordinates (e.g. only a few voxels or a few stocks undergo a change in data generating mechanism), and the aim is to borrow strength across the different components to detect smaller changes than would be possible through only seeing any one of the individual series.

The large majority of the work in this area has focused on the ‘offline’ problem, where one sees the entire data set in advance of trying to ascertain where change(s) occurred. However, the corresponding online problem, where one sees the data coming in over time and seeks to declare a change as soon as possible after it has occurred (with a small probability of false alarm), is at least as important from the point of view of applications. This essay would review existing work, but offers considerable scope for the candidate to propose and study a new method for high-dimensional online changepoint detection.

Relevant Courses

*Useful: Modern Statistical Methods, Topics in Statistical Theory*

References


41. Minimax Lower Bounds in Modern Statistical Problems  
Professor R. J. Samworth

While much of statistical theory concentrates on analysing the performance of particular statistical procedures, there is also great interest in understanding the information-theoretic limits achievable by any method. In this sense, minimax lower bounds provide a way of characterising the fundamental difficulty of a statistical problem.

This essay should begin with a brief description of the basic ideas, e.g. Assouad’s and Fano’s lemmas ([1]). The use of minimax lower bounds in one or more examples of modern statistical problems should then be illustrated in depth. Possible topics include shape-constrained estimation problems ([2]), sparse linear regression ([3]), community detection in network analysis ([4]), or potentially a new problem proposed by the candidate.

Very recently, researchers have started to consider whether the minimax rate may change under additional constraints, which may be related to, e.g., computation ([5]), privacy ([6]) or communication ([7]). An ambitious candidate might like to cover one or more of these topics too.

Relevant Courses

Useful: Topics in Statistical Theory, Modern Statistical Methods, Nonparametric inference under shape constraints

References


42. The Bootstrap in High Dimensions  
Professor R. J. Samworth

The bootstrap ([1]), together with its variants such as bagging ([2], [3], [4]), is one of the most important ideas introduced into Statistics in the last 40 years. After intensive efforts in the 1980s and 1990s, its properties are reasonably well understood in low-dimensional contexts. However, it is only recently ([5]), that there has been significant progress in understanding
bootstrap approximations to maxima of sums of high-dimensional random vectors, and in high-dimensional regression ([6]) and Principal Component Analysis ([7]). This is now a very active topic of research.

There are several possible directions for an interested candidate to explore. One would be to focus on the theoretical contributions of [5], together with subsequent developments, as well as [6] and [7]. Another would be to explore theoretically and/or empirically the performance of bootstrap methods in different high-dimensional statistical models (e.g. Cox models), building on recent work in high-dimensional linear models in [8], which studies the bootstrap applied to a debiased Lasso estimator.

Relevant Courses

Useful: Modern Statistical Methods, Topics in Statistical Theory

References


43. The Data Science of Imaging – From Computation to Information ...... Professor J. Aston and Dr Carola-Bibiane Schönlieb

Image analysis consists of many steps which start at the acquisition of the raw data and end with a fully reconstructed, motion corrected, registered and segmented image. Each of these steps consists of many possible algorithms, with different uncertainties attached to their results. However, in almost all cases, the analysis proceeds linearly from one step to another, without taking into account the fact that errors are propagated from step to step. Modern Data Science recognises that it is necessary to incorporate ideas from both the computational analysis side as well as the statistical analysis perspective.

This essay will look at a hybrid approach for modelling of such issues for the particular case of image registration, with elements from both computational analysis and statistics. Image registration denotes the task of aligning images to one coordinate system. From the statistical side, in [1,2] metric based procedures for image registration are incorporated in a subsequent
statistical modelling for a population of images, such as medical brain scans or photographs of human faces. The link between such a statistical analysis and the transformation that the images have undergone when registered to a template, is crucial for determining principal separating features (PCA) within a population of images, quantifying deviation from an ‘average’ image, such as a healthy brain scan, or to cluster images in different classes, such as detecting the face of the same person in different photographs. While existing hybrid models in statistics are effective in capturing the change in the statistical distribution within such a population, metric-based registration methods that can be accommodated in such approaches only form a small selection amongst the state of the art for image registration. Very powerful image analysis, and in particular image registration techniques, arise in a variational regularisation framework, acknowledging the fact that image registration is a highly ill-posed problem [3]. Physical models for registration functions can be used, encoding for instance whether a registration is rigid or elastic in the form of an appropriate regularisation [4], and whether a registration is mass-preserving or viscous [5,6].

The essay could investigate a number of techniques and determine their ease or difficulty in being combined into joint analyses. A dominant difficulty in hybrid models is the complexity, often resulting into nonlinear and non-convex variational problems whose analysis and robust computation are challenging. A more theoretically oriented essay could study the well-posedness and the statistical validity of a joint analysis. Computational requirements are, of course, paramount so an essay could investigate these. Alternatively a more empirical essay could examine the role of errors in affecting a proposed joint analysis scheme from the literature from both a computational and statistical perspective.

Relevant Courses

Essential: Modern Statistical Methods, Inverse Problems in Imaging.
Useful:

References


44. The Mathematics of Precision Medicine

Professor J. A. D. Aston

While there are many definitions of Precision Medicine, the most common is that of using data to inform medical diagnosis and treatment specific to individuals, precisely targeting what is best for that individual. However this requires data from many individuals to be considered. Data is often very high dimensional, including genetic, imaging and biomedical test results along with other demographic information. Condensing this information into something that can be used in practice, without releasing sensitive medical records from others, requires mathematical and statistical techniques such as those considered in high dimensional statistical learning [1] and differential privacy [2].

There is considerable scope in this essay. A theoretical essay could examine the interplay between differential privacy and statistics, reviewing literature at the intersection, and possibly suggesting new ideas to keep high dimensional data private. A more applied essay could consider data from one of the large repositories such as UK Biobank, and investigate practical techniques to recover information from the data.

Relevant Courses

Useful: Modern Statistical Methods, Coding and Cryptography (Part II)

References


45. The Statistics of Manifold Data

Professor J. A. D. Aston

Many observed data are constrained by their intrinsic features or geometry. This is especially true when the objects under analysis are shapes or images, as in many cases the angle of view and the magnification of the shape is unimportant [1]. This is also linked to the study of data which arises in particularly spaces, such as data observed on a (hyper-)sphere [2,3], or where the observations are types of matrices, such as those which are positive definite [4]. All these settings yield data that are inherently non-Euclidean, but most statistical analysis is predicated on the data coming from a Euclidean space.

The idea of this essay will be to review some of the recent advances in shape and related statistics, many of which are based on concepts from differential geometry. There is then considerable scope in the essay. Theoretical investigation could be undertaken into some of the underlying metrics that are used in statistical shape analysis. Methodology for certain special cases of shape data could be compared, such as different methods for the statistical analysis of samples of positive definite matrices. Alternatively, data analysis could be undertaken for shape observations derived from images, for example.
Relevant Courses

Useful: Differential Geometry, Modern Statistical Methods, Topics in Statistical Theory

References


46. Convergence of Markov Processes .................................

Professor J. R. Norris

When a random process has a Markov structure, this allows an economical description of its dynamics. This structure also provides tools to handle limit operations, such as are of interest in stochastic models for large populations or in large random structures.

This essay is intended to offer a flexible opportunity to write an account describing some aspect of the theory or applications of the convergence of Markov processes, to be agreed.

One approach would be to set out in some generality some of the methods available to show convergence, where the limit objects are either differential equations, or diffusion processes, or continuous-time Markov chains, or some combination of these. Then this general theory should be illustrated by discussion of examples, perhaps with a coherent theme, such as models from mathematical biology, or large random structures. The encyclopedic text of Ethier and Kurtz [1] can be used as a starting point.

A second approach to this essay would be to focus from the outset on a single harder example, such as convergence of the discrete Gaussian free field to a continuum limit, or convergence of some stochastic model of interacting particle dynamics in the limit of large numbers.

Relevant Courses

Essential:

Useful: Advanced Probability, Stochastic Differential Equations

References


47. Sufficient Conditions for Singleton-transitive Compositional Graphoids .

Dr K. Sadeghi

In graphical models and causal inference, nodes of the graph correspond to random variables and missing edges to conditional independencies. The connection between graphs and probability
distributions is usually established in the literature by the concept of (global) Markov property, which ensures that if node $i$ is separated from node $j$ of the graph by the node subset $C$ then random variables $X_i$ and $X_j$ are conditionally independent given the random vector $X_C$ in the probability distribution [2]. In addition to this there exists a so-called pairwise Markov property which provides an independence statements for every missing edge in the graph. Pairwise and global Markov properties are equivalent for undirected graphs under a set of axioms for probability distributions, called graphoids [5,2].

Recently this equivalence has been established for other types of graphs, under compositional graphoid axioms [3]. In order for the other direction of global Markov property to hold, namely every conditional independence in the probability distribution to correspond to a separation in the graph, the additional axiom of singleton-transitivity is also needed [1].

The goal of this essay is to review the results in the literature for what distributions satisfy these properties or what sufficient conditions distributions must satisfy in order to satisfy these properties. Later on, one can try to come up with new sufficient conditions based on the common features of such distributions, or algorithms to test these properties from data.

**Relevant Courses**

*Useful:* Modern Statistical Methods

**References**


48. Arm Exponents in Critical Percolation and SLE

*Professor N. Berestycki*

One of the extraordinary achievements which was made possible by the introduction and the development of the Schramm-Loewner Evolution (SLE) theory is the rigorous computation of some “critical exponents” which describe the universal large-scale behaviour of models from statistical physics such as percolation at criticality, in two dimensions.

The aim of this essay is to discuss a rigorous of the computation of the “one-arm” and “two-arms” exponents in percolation, through SLE techniques. The one-arm exponent is defined as follows: consider critical percolation in the plane, and let $p_n$ be the probability that the origin is connected to a point at distance $n$. Then the one-arm exponent is the number $\nu_1$ such that $p_n = n^{-\nu_1 + o(1)}$. Likewise, the two-arm exponent $\nu_2$ concerns the limiting probability that the origin is connected by an open path and by a closed path to distance $n$. The goal of the essay will be to present the computation of these exponents in the continuum using SLE$_6$ (they are respectively equal to $5/48$ and $1/4$) and to explain how this can be used to compute the discrete
percolation exponents. Depending on interest, further questions can be investigated (other kind of exponents which can be computed from SLE, the difficulty with monochromatic exponents, etc.)

References


49. Cutoff for the Ising model and noisy voter ..............................

Professor N. Berestycki

The Glauber dynamics for the Ising model is a simple and natural Markov chain whose stationary distribution is the Ising model. At each step a vertex is chosen at random on the graph, and its spin is updated by choosing its distribution from the Ising model given the neighbours. Understanding the mixing time and showing the cutoff phenomenon for this chain on $\mathbb{Z}^d$ (at high temperatures, i.e., for $\beta < \beta_c$) was a longstanding problem which has recently been resolved by Lubetzky and Sly, using a new method called *information percolation*.

Another similar model is the noisy voter model, where vertices of a graph are assigned an opinion (0 or 1). At each step, a randomly chosen vertex changes its opinion to that of one of its neighbours chosen at random, or, with fixed probability, chooses its opinion at random. In fact it turns out that on the $n$-cycle, the noisy voter model and the Ising models are exactly identical. However, cutoff for the noisy voter model in general was established in [2] using very different methods than for the Ising model in [1].

The goal of this essay will be to give a brief description of the Ising model and its phase transition, as well an overview of the proof of cutoff by Lubetzky and Sly using information percolation (an excellent overview of the proof, at least for very low temperatures, can be found in [3]). The essay would also explore the noisy voter model [2] and whether ideas of [3] can be used for the proof of cutoff for the noisy voter model.

References


50. Graphical Modelling for High-Dimensional Data ..........................

Dr K. Sadeghi and Dr R. D. Shah

When faced with high-dimensional data, it is often of interest to understand relationships between the variables. Whilst the basic neighbourhood selection technique of [1] and the graphical
Lasso [2] have been successful in many applications, there remain many challenges concerning their use when data is non-Gaussian, as is often the case. Attempting to estimate the DAG associated with the data distribution by using the PC-algorithm [3] for example, presents even more difficulties, and even when data is in fact Gaussian the procedure may not be uniformly consistent [4].

This essay would review some of the recently proposed approaches to address the challenges above ([5-8], for example) and study empirically and / or theoretically their performance. Depending on the interests of the student, focus can be primarily on estimation of the conditional independence graph, or can be directed largely towards the inconsistency and sensitivity of the PC-algorithm in the non-Gaussian and high-dimensional setting and other approaches to estimate the DAG. There is some scope for proposing modifications to existing methodology, though this is not a requirement.

Relevant Courses

*Essential: Modern Statistical Methods*

References


51. Randomised Algorithms for Large-scale Data Analysis ........................

Dr R. D. Shah

The size of modern datasets presents enormous computational challenges for statistical methods, and developing scalable methods is currently the topic of much research activity across statistics and machine learning. A general approach that has proved successful for a variety of large-scale data problems is to use some form of randomisation within algorithms applied to the data.

Given an $n \times p$ data matrix $X$ with rows as observations, one line of work has considered randomly mapping $X \rightarrow S$ where $S$ is $m \times p$ with $m \ll n$, and studies the performance of algorithms applied to this compressed or ‘sketched’ data $S$ [1, 2]. A related approach involves
compressing the columns of $X$ rather than the rows [3]. Another direction of research concerns developing algorithms that only work with small randomly selected subsets of the data at any one time [4, 5].

There are a number of routes this essay could take. One option is to review some of the different approaches outlined above and compare them theoretically and / or empirically. Alternatively, a student could focus largely on one particular line of work and aim to modify and extend existing methodology.

**Relevant Courses**

*Essential:* Modern Statistical Methods  
*Useful:* Topics in Convex Optimisation

**References**


52. **Cover Times and the Discrete Gaussian Free Field** .......................  
Dr P. Sousi

Consider a simple random walk on a finite connected graph $G = (V, E)$ started from a given vertex $v_0$ and wait until the first time that all vertices have been visited at least once. The expectation of this time, which we call the cover time, is a fundamental parameter of $G$.

The discrete Gaussian free field (DGFF) $\{h_v\}_{v \in V}$ with $h_{v_0} = 0$ on the graph $G$ is a centred Gaussian process with covariances given by effective resistances. The DGFF and its continuous counterpart play a fundamental role in the theory of random surfaces.

A beautiful connection between these two objects was exhibited in a recent breakthrough paper by Ding, Lee and Peres [1]. They established that the cover time of any graph is up to constants the product of the number of edges of the graph and the square of the expectation of the maximum of the DGFF.

They conjectured that when the cover time is much bigger than the maximum hitting time of the graph, then the two quantities should be related up to a $1 + o(1)$ term, where the limit is taken as the size of the graph tends to infinity. In [1] the upper bound was proven, but a couple of years later, Alex Zhai [2] was able to prove the lower bound under the same assumption.

A successful essay will contain the proof of the above equivalence and all of the necessary background.
Relevant Courses

Useful: Advanced Probability

References


53. Online Principal Component Analysis

Dr Q. Berthet

Principal component analysis (PCA) is an ubiquitous technique in data analysis, used to reduce the dimension of a dataset and increase its interpretability. It can also be used in statistics to estimate vectors related to the covariance matrix $\Sigma$ of an unknown data-generating distribution, by applying it to the empirical covariance matrix $\hat{\Sigma}$ of a dataset $X_1, \ldots, X_n$.

In modern applications of statistics, the data often comes in the form of a continuous stream, rather than a fixed observation, and it is often of very large size. This makes it prohibitive to compute a new empirical matrix and its leading eigenvectors, for every new sample. Online PCA addresses this issue by only keeping partial information at every step, and to do a simple update of the estimated principal components.

A successful essay would present the known techniques on this problem and investigate how these can be applied to problems in statistics, in particular for network analysis.

Relevant Courses

Essential: Modern Statistical Methods.

Useful: Topics in Convex Optimisation, Topics in Statistical Theory.

References


54. Linear Regression on Graphs .................................................. Dr Q. Berthet

An increasing number of problems in data science involve complex data structures. As an example, it is often possible to incorporate in a statistics problem information coming from a network. This essay would focus on the problem of estimating a signal $\theta^* \in \mathbb{R}^n$ where each coefficient is associated to one of the $n$ vertices of a known graph $G$. This can be done by observing noisy linear measurements $X\theta^* + \varepsilon$, and using structural assumptions on how the signal is related to the graph.

This problem, and the techniques involved to solve it, are related to other problems in high-dimensional statistics, such as using the Lasso estimator to solve a sparse linear regression problem. A successful essay would describe one or several of these problems as treated in the literature, and introduce some new developments, on the aspects related to statistics or to optimisation.

Relevant Courses

Essential: Modern Statistical Methods.
Useful: Topics in Convex Optimisation, Topics in Statistical Theory.

References


55. Structure Formation in Saturn’s Rings .......................................... Dr H. N. Latter

The rings of Saturn are perhaps the most familiar and beautiful objects in astrophysics; they also exhibit some of its most puzzling phenomena ([1]). Complex patterns and waves have been carved into the disk by a myriad of processes: viscous and gravitational instabilities, impacts with micrometeoroids, and the gravitational influence of external and internal moons, to name but a few. The patterns extend over a vast range of lengthscales (from 100 metres to 100
kilometres) and are only partially understood. For a review of their observations see [2], for a review of their theory see [3], and for a general review of planetary rings see [4].

Your essay could either (a) survey the observations of these structures and the relevant physics in each case, or (b) concentrate on just one class of structure and go into some mathematical/physical detail. Specific topics that could be discussed include:

(i) How to model the rings. The kinetic theory of cold and dense granular flow.
(ii) Gravitational instability and canted self-gravity wakes in the A and B-rings.
(iii) Viscous overstability and periodic microstructure on 100m scales in the A and B-rings.
(iv) The ballistic transport process: sharp inner ring edges, and 100-1000km structure in the C-ring and inner B-ring.
(v) Spiral density wave launching in the A-ring by external satellites.
(vi) Embedded 100m moonlets in the A-ring (‘propellers’).
(vii) The bizarre dynamics and structures of the braided F-ring.

Relevant Courses

Useful: Astrophysical fluid dynamics, Dynamics of astrophysical disks

References


56. Chiral Symmetry and Skyrmions ............................

Professor N.S. Manton

SU(2) isospin symmetry is an accurate and useful symmetry of hadronic physics, but it is believed to be a subgroup of an approximate SO(4) chiral symmetry that is spontaneously broken. The pions are the Goldstone bosons of this broken symmetry, for which the main physical evidence is that they are much less massive than nucleons or other mesons. Chiral symmetry can be analysed either as an approximate symmetry of QCD, or using more phenomenological field theories known as nonlinear sigma models.

In this essay you should summarise how approximate chiral symmetry arises in QCD, but the main part should discuss the consequences of the symmetry in pion-pion interactions, in pion-nucleon interactions, and also how it can be used to constrain models of the nucleon-nucleon force. You may also briefly discuss Skyrmions, the solitons in nonlinear sigma models, and their interpretation as nucleons.
Relevant Courses

Useful: Quantum Field Theory; Symmetries, Fields and Particles; Standard Model

References

[6] N. Manton and P. Sutcliffe, Topological Solitons, Cambridge University Press, 2004 (Chapter 9 on Skyrmions, see also Donoghue et al.).

57. The Theta Angle in Gauge Theories

Professor D. Tong

In both Maxwell and Yang-Mills theories, there is an extra term that you can add to the Lagrangian compatible with Lorentz symmetry. It takes the form $\varepsilon^{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma}$ and is called the “theta term”. It is a total derivative but nonetheless plays an important physical role. The purpose of this essay is to describe the theta term, its properties and some of its applications.

In the context of Maxwell theory, the theta term provides an effective description for a class of materials known as “topological insulators”. The theta term gives a simple explanation for the response of these materials to electric and magnetic fields. It also reveals the importance of magnetic monopoles as a way to characterise these materials, using the Witten effect and other magneto-electric phenomena.

The theta term plays a more prominent role in Yang-Mills theory. It changes the vacuum structure, spectrum and dynamics of the theory. These changes can be seen through the subtle, topological manner in which the theta term affects canonical quantisation. It also shows up more directly in the path integral through instanton physics. The “strong CP problem” is the question of why the theta angle appears to vanish in QCD: it remains unsolved.

Relevant Courses

Essential: Quantum Field Theory; Advanced Quantum Field Theory.
Useful: Symmetries, Fields and Particles; The Standard Model.
References

[1] Some basic properties of Maxwell theory with a theta term were described by F. Wilczek in “Two applications of axion electrodynamics”, Phys. Rev. Lett. 58, 1799 (1987)


58. The First Law of Black Hole Mechanics

Professor H.S. Reall

In the early 1970s, it was shown that black holes obey a set of classical laws, known as the laws of black hole mechanics, which are identical in form with the laws of thermodynamics. Hawking’s discovery of black hole radiation revealed the underlying reason for this.

The first law of black hole mechanics concerns small perturbations of a time-independent black hole. It relates the change in the area of the event horizon to the change in the mass and angular momentum. The original proof of the first law of black hole mechanics [1] encompassed only perturbations to other time-independent, axisymmetric, black holes. More recently, Sudarsky and Wald obtained an alternative proof that allows for arbitrary perturbations [2] (see [3] for a simplified account).

In other work, Wald and collaborators have obtained versions of the first law that apply in any diffeomorphism-invariant theory of gravity [4,5], for example higher-derivative modifications of General Relativity such as those arising in the classical limit of string theory. This leads to a definition of black hole entropy for such theories. This has played an important role in recent discussions of black holes in string theory.

The essay should give a detailed explanation of Sudarsky and Wald’s proof of the first law of black hole mechanics. Then it should describe the derivation of the first law of black hole mechanics for general diffeomorphism-invariant theories. The essay should be written at a level that would be understood by another Part III student who had attended the Black Holes course.

Relevant Courses

Essential: General Relativity, Black Holes.

Useful: Quantum Field Theory.
References


59. Axions ................................................................. Professor B.C Allanach

Peccei-Quinn symmetry was introduced to solve the strong $CP$–problem. Such a solution predicts a new particle: the axion, which could also potentially constitute the dark matter of our universe.

The essay should try to follow analytic calculations in some detail and present them clearly. The essay should be in two parts: the first half of the essay summarising the motivation for axions, their phenomenology and constraints upon them. The second half should then examine a more in-depth topic, for example: relaxions or chameleons.

Relevant Courses

Essential: Quantum Field Theory, Standard Model, Particles and Symmetries, Advanced Quantum Field Theory

References

[1] Part of the point of this essay is for you to develop the literature search skills. Start by googling axions and see where it leads you!

60. Two Dimensional Yang-Mills Theory .................................................. Dr. D.B. Skinner

In four space-time dimensions, Yang-Mills theory is a typically intractable, strongly coupled quantum field theory. But in two dimensions, life is much better: One can analytically compute not just the Yang-Mills partition function, but also all correlation functions of insertions of Wilson loops. The techniques involved are very different from the standard, perturbative methods one learns about in the Part III QFT lectures, and range from lattice gauge theory to non-Abelian equivariant localization. This essay will explore one or more approach to understanding 2d Yang-Mills as a quantum theory.

Relevant Courses

Essential: Quantum Field Theory, Advanced Quantum Field Theory.
Useful: Lie Algebras and their Representations, Algebraic Topology.
Clifford operations are an important class of quantum gates that have many applications across quantum computing, and they have both elegant mathematical properties and important physical implementational features. The Gottesman-Knill theorem asserts that Clifford circuits acting on computational basis inputs can be classically efficiently simulated; hence they offer no computational benefit over classical computing. However by including further (even seemingly innocuous) resources it is possible to regain full universal quantum computing power.

In this essay we'll explore some aspects of the computational power of (extended) Clifford circuits. As a starting point, the Gottesman-Knill theorem (cf [1] and [5]) should be stated, and proved using the stabiliser formalism (which will require a summary account of the formalism itself, see e.g. [1] §10.5, and [6] for this and further issues). Then a further topic related to the computational power of Clifford circuits including some extra non-Clifford resources should be discussed. This could be drawn from any one (or more) of the following possibilities:

(a) in [2] the classical simulation complexity of Clifford circuits with a variety of extra resources is considered (many of which cannot be represented in the stabiliser formalism);
(b) in [3] the notion of a “magic state” is introduced, and their inclusion into Clifford computations regains full quantum computing power. This topic also requires the notion of a density matrix or mixed state (cf [1]), and uses an application of the stabiliser formalism [1,6] for a process called magic state distillation.
(c) in [4] the classical simulation complexity of Clifford circuits that are allowed to contain also a restricted number of T gates is considered. (Clifford circuits with unrestricted use of T gates is again quantum universal). It is based on the novel interesting notion of stabiliser rank of an n-qubit state.

Further possibilities from the literature could also be considered if desired.

Relevant Courses

*Essential:* Quantum computation

References

Professor C. P. Caulfield

It is of great practical importance to understand how flows undergo the transition to turbulence, as turbulence typically hugely increases mixing, transport and dissipation within flows of environmental and industrial interest. It is commonly believed that ‘normal’ mode flow instabilities play a central role in such transition processes, and the conventional argument is that the ‘most unstable’ normal mode will dominate the nonlinear evolution of the flow, and hence lead the flow to transition. However, the underlying linearized operator is non-normal, and so it is possible for substantial transient growth of perturbations to occur. [1] A particularly attractive method to consider such transient growth problems is the so-called ‘direct-adjoint looping’ method, which can be generalised to consider fully nonlinear perturbations, where the developing perturbations can reach a sufficiently large amplitude to nontrivially modify the ‘base flow’. [2] This method is particularly well-suited to consider generalised stability problems, where the measure which is being extremised is not necessarily the ‘energy’ of the developing perturbation. [3] Indeed, there are several interesting mathematical issues about the most appropriate measures to use, [4] and this essay could approach the general issue of perturbation ‘growth’ in a range of environmentally and industrially important flows from a variety of mathematical and computational directions.

Relevant Courses

Essential:
Hydrodynamic stability.

References

63. Exotic Hadrons .................................................................

Dr C. E. Thomas

Historically, the pattern of hadrons could be understood in a phenomenological model where a meson consists of a 'constituent' quark-antiquark pair and a baryon consists of three constituent quarks. However, over the last decade experiments have observed a number of hadrons which do not fit into this simple picture. Many possible explanations for these so-called 'exotic hadrons' have been put forward including that some are hybrid hadrons (where the gluonic field is excited) or tetraquarks (containing two quarks and two antiquarks). The essay should give a short overview of the theoretical and experimental situation, and then focus on either hybrid hadrons or tetraquarks. A discussion of the expected pattern(s) of flavour, spin, parity and charge-conjugation-parity quantum numbers should be included.

Relevant Courses

Useful: Symmetries, Fields and Particles; The Standard Model

References


64. Interacting Internal Waves ......................................................

Professor S. B. Dalziel

Internal gravity waves [1] play an important role in both the atmosphere and the ocean, providing both a mechanism for the transport of energy and momentum, and a mechanism for driving mixing at locations remote to where the energy is input. There is increasing recognition of the role played by resonant nonlinear interactions between waves of different frequencies in driving waves towards the point of breaking through mechanisms such as parametric subharmonic instability [2]. Most existing work considers the quasi-steady interaction of plane waves or wave beams where the locations and characteristics of the two beams are kept constant. The purpose of this essay is to explore the case where the frequency of one of the waves is changing over a time scale that is not long compared with the growth of the instability.

An essay on this topic would combine a literature review with some illustrative laboratory experiments. The review should include salient aspects of internal gravity wave theory and parametric subharmonic instability for quasi-steady interactions, focusing on cases where the interaction is between two beams of finite width. The experimental component would first demonstrate the existence of the instability over some range of interaction parameters, then explore how the interactions are changed by sweeping through the range of valid frequencies.
of one (or both) of the incoming waves. This experimental component would make use of a recently developed device capable of producing internal gravity waves with an arbitrary spectral evolution.

Relevant Courses

Essential:
Under graduate fluid mechanics.

Useful:
Fluid Dynamics of the Environment

References


65. Do Fish Mix the Ocean? .......................................................... Professor S. B. Dalziel

Mixing of the density stratification plays an important role in driving the circulation of the oceans [1]. Dominant mechanisms derive their energy from the tides, winds or radiative heating and cooling, but there are other sources. One such mechanism is ‘biogenic mixing’, the mixing generated by biological activity within the oceans [2].

An essay on this topic would concentrate on the potential for fish to mix a stratified fluid, focusing on the physics rather than the biology of the problem. The essay would begin by reviewing the available literature on mixing due to fish and marine mammals, describing the mechanisms that could contribute to mixing. The essay could then proceed to either draw on the existing literature for the ‘mixing efficiency’ [3] of different flows to assess the potential for mixing at a pycnocline, or to conduct some simple laboratory experiments using battery-powered fish to visualise and quantify the mixing produced.

Relevant Courses

Essential:
Under graduate fluid mechanics.

Useful:
Fluid dynamics of the environment.

References


66. Separating Fluids by Faraday Waves ............................................ Professor S. B. Dalziel

Faraday waves are can be thought of as a subharmonic instability of a fluid surface subject to a periodic vertical acceleration. For small forcing amplitude, the fluid surface remains flat, but as the amplitude is increased beyond a critical value, a standing wave pattern forms. For strong forcing, the pattern can be described as an array of ‘jets’ that pinch off through a capillary mechanism to form discrete droplets [1,2]. Using ultrasonic transducers to provide the forcing at very high frequencies provides a technique for producing very small droplets, an approach that has many industrial uses.

This essay will address the question of whether the droplet formation possible through Faraday instability provides a potential mechanism for separating mixture of fluids into its different constituents. The essay would begin with a brief literature review of Faraday waves and droplet formation before looking into the differences seen if the fluid contains surfactants. As the surface area of the droplets produced will soon become substantially greater than that of the fluid layer being forced, a key question is whether the surfactant concentration in the fluid layer is significantly reduced in the process. There is scope within this topic for some simple laboratory experiments, both to explore the properties of Faraday waves and droplet formation, and to assess the impact of surfactants.

Relevant Courses

Essential:
Undergraduate fluid mechanics.

Useful:

References


67. Lifts of Polytopes ................................................................. Dr. H. Fawzi

A polytope is the convex hull of a finite number of points. A lift of a polytope is a representation of that polytope, call it $P$, as the projection of another higher-dimensional polytope $Q$ which is “simpler” – typically we want $Q$ to have much fewer facets than $P$. The idea of lifting plays a crucial role in mathematical optimisation since the existence of a “small” lift means we can efficiently solve optimisation problems over $P$. Good introductions to the idea of lifts are [1] and [2].
There has been a lot of progress recently in understanding when a polytope has (or not) a small lift. This essay will discuss some of the recent developments in this topic and its connection to some matrix factorisation problems [2]. A possible direction also is to investigate certain particular polytopes, like e.g., cyclic polytopes: these polytopes play an important role in polyhedral combinatorics but their extension complexity (size of smallest lift) is not yet known; see [3,4] for more on these polytopes.

**Relevant Courses**

*Useful: Topics in Convex Optimisation*

Knowledge of convex analysis and convex geometry would be useful.

**References**


**68. Wave Attractors in Rotating and Stratified Fluids**

Professor G. Ogilvie

Internal waves can propagate in rotating and/or stably stratified fluids (as often occur in astrophysical and geophysical settings) as a result of Coriolis and/or buoyancy forces. Their properties are radically different from those of acoustic or electromagnetic waves. The frequency of an internal wave depends on the direction of the wavevector but not on its magnitude. Waves of a given frequency follow characteristic paths through the fluid and reflect from its boundaries. In many cases the rays typically converge towards limit cycles known as wave attractors. One application of this finding is to tidally forced fluids in astrophysical and geophysical settings. If tidal disturbances are focused towards a wave attractor, this can lead to efficient tidal dissipation that in some cases is independent of the small-scale diffusive processes.

This essay should review the subject of internal wave attractors, including some of the more recent developments. Some simple explicit examples should be provided, which could involve original calculations. Topics which might be covered include:

1. The behaviour of rays for pure inertial waves in a uniformly rotating spherical shell.
2. The relation, if any, between the propagation of rays within a container and the existence of inviscid normal modes.
3. The consequences of a wave attractor for the decay rate of a free oscillation mode, or the dissipation rate of a forced disturbance, in the presence of a small viscosity.
4. The roles of nonlinearity and instability in wave attractors.
5. The relevance of wave attractors to tidal dissipation in astrophysical systems.
Relevant Courses

Useful: Astrophysical Fluid Dynamics

References


Dr D. R. Hewitt

Viscoplastic fluids are a branch of non-Newtonian fluids characterised by a plastic ‘yield stress’, below which they behave as a rigid solid, and above which they flow like a viscous fluid. A vast array of materials, from mud and industrial waste to toothpaste and ketchup, exhibit some degree of viscoplasticity, and can behave in quite a different manner to their Newtonian counterparts. Flow of such fluids in thin conduits is widespread in industry and geophysics, including in the squeeze flow of industrial waste to remove water, the transport of slurries down thin hydro-fractures, and the flow of liquefied mud through underground conduits after an earthquake.

This essay will review lubrication theory for viscoplastic fluids, starting with simple ‘Bingham’ fluids, and discuss models for both squeeze flow and parallel flow in a slot. The essay should address the so-called ‘lubrication paradox’, and discuss the flow structures and relevant forces required to drive flow. Beyond this core review material, the essay may take slightly different directions depending on the choices of the candidate; some suggestions for investigation include the effect of topography or sloping walls in the slot, the ‘plastic’ limit and links to classical plasticity theory, or the extension to more complex rheological models.

Relevant Courses

Essential:
Slow Viscous Flow

Useful:
Perturbation Methods Fluid Mechanics of the Solid Earth

References

Modern cameras capture images of millions of pixels of ever increasing quality. Although the image quality is getting better and better there is still need to process those images by mathematical algorithms to reduce artifacts like noise and blur due to low light, motion, etc.

Many of those mathematical algorithms to enhance image quality minimize a cost functional where we trade-off the closeness to the acquired image data with the regularity that we expect in natural scenes. For natural images it is clear that the image that we are really after does not contain random variations like noise but often contains a few distinct objects. A popular model to capture such image regularity is the total variation functional (and its variants) which penalizes variations in the image due to noise but at the same time allows major edges in the image to be kept. While being advantageous in terms of modelling, the total variation functional is non-smooth which makes the minimization more difficult.

Despite non-smooth variational methods (minimizations of cost functionals) being popular in many scientific fields, their solution is still challenging, particularly when the number of unknowns is very large. In recent years the branch of stochastic optimization gained more and more interest for large scale applications like machine learning. The basic idea is that at any point in the algorithm not the whole data set is being used but only a smaller subset which can be better handled. As the mathematical models in machine learning and imaging are largely the same, this essay will explore computational methods from stochastic optimization for their use in imaging.

The essay writer is expected to understand some models that are used for image denoising and some stochastic optimization algorithms that can solve the relevant minimization problem. As a computational/applied essay it is important that the essay writer has some knowledge about scripting languages (either MATLAB or python).

**Relevant Courses**

*Essential: -*

*Useful: Inverse Problems in Imaging*

**References**


The remarkable phenomenon of the atmospheric QBO

Dr A. D. Ming

The quasi-biennial oscillation (QBO) is a pattern of alternating eastward and westward wind features that descend through equatorial stratosphere at about 1 km per month from about 30 km down to about 16 km. The whole cycle takes about 28 months to repeat and is an important component of the variability of the stratosphere. An introduction to the dynamics of the QBO can be found in Baldwin (2001), Section 3 and in Plumb (1977). Theories explaining the QBO need to address (1) the periodicity, (2) the occurrence of zonally symmetric eastward flow over the equator and (3) the downward propagation without loss of amplitude and more recently (4) why the QBO skipped a beat in 2016 with a westward jet forming inside an eastward phase. The essay should provide as first part a review of the dynamics of the QBO describing the two-way interaction between waves and the zonal mean flow. It should then move on to address one or two other topics such as the influence of the QBO on other regions of the stratosphere, the numerical modelling of the QBO, the recent 2016 disruption or the QBO as an indicator of possible changes in the stratospheric Brewer-Dobson circulation. The essay could be supplemented by the analysis of an observational climate dataset. Interested candidates should contact Alison Ming for further advice.

Relevant Courses

Essential: Undergraduate fluid dynamics

References


Deep Variational Models

Dr M. Benning, Dr C.-B. Schönlieb

One of the most successful approaches to approximate solutions of inverse problems in imaging is to cast the approximate solution as the minimiser of a variational model. The key to success of such an approach is to define a variational energy such that its minimiser reflects structural properties of the imaging problem. For a long time, these energies were merely handcrafted models, such as the total variation...
regularisation [1] and its extensions [2]. Variational models constitute deterministic and mathematically rigorous models with stability and approximation guarantees as well as a control on qualitative and physical properties of the solution. On the downside, these methods are rigid in the sense that they can be adapted to data only to a limited extent. The total variation regularisation, for instance, is effective in removing noise from piecewise constant images but is usually too simple to capture the complex structural properties of natural or medical images. Convolutional neural networks in machine learning, on the other hand, offer powerful tools for adaptively analysing and processing data – however, often hiding intrinsic features of the solution mechanism and often with a lack of theoretical guarantees, e.g. stability to noise.

As a consequence, researchers started to apply machine learning techniques to ‘learn’ more expressive variational models [3,4]. The basic principle is to consider a bilevel optimisation problem, where the variational model appears as the lower-level optimisation problem and the higher-level optimisation problem is represented by a loss function that measures the reconstruction error between the solution of the variational model and given ground truth reconstructions. This essay should explore the idea of bilevel learning [5], in particular by studying bilevel learning of reaction-diffusion models [6]. Depending on the interest and ambition of the essay writer, a more applied essay could investigate these approaches in the context of different imaging tasks, such as Magnetic Resonance Imaging from incomplete data, or limited-angle Computerised Tomography. Such a computationally focused essay could investigate the challenges in the optimisation and possible novel pathways for efficiently solving bilevel optimisation problems. A more theoretically oriented essay could study the relationship of the discretised reaction-diffusion models and their corresponding partial differential equations in the continuum, as well as the relationship of bilevel learning to convolutional neural networks.

**Relevant Courses**


**References**


73. The HHL Algorithm .................................................. Dr S. Strelchuk

One of the most common problems in the sciences and engineering is to find a vector $x$ satisfying $Ax = b$ for a given matrix $A$ and vector $b$. When $A$ is an $n \times n$ matrix the best known classical methods for solving such linear systems have runtime proportional to $n$.

In 2009 Harrow, Hassidim and Lloyd (HHL) discovered a remarkable quantum algorithm for solving linear systems of equations having runtime $\text{poly}(\log n)$ and favourable scaling in other parameters such as precision when $A$ is sparse and Hermitian [1]. In addition, they showed that an arbitrary quantum computation can be recast as a problem of this kind. The HHL algorithm incorporates the use of quantum Hamiltonian simulation and phase estimation. It has many applications ranging from solving linear differential equations [2] and boundary-value problems [3] to quantum data fitting [4] and machine learning [5]. In a recent paper, Childs et al. gave a version of the HHL algorithm with an exponentially improved dependence on precision [6].

The aim of this essay is to provide an exposition of the algorithm and possibly one or more of its applications if desired.

Essential Courses

Part III Quantum Computation (M16).

References


74. Asymptotics Beyond All Orders ................................. Dr S.J. Cowley

Background

In a number of problems in widely different contexts it has been discovered that determining one or more of the first ‘exponentially small’ terms, beyond the usual infinity of algebraic ones, holds the key to understanding some crucial aspects of the problem’s solution. Examples include tunnelling problems in classical and quantum barrier penetration, determination of the reflected wave in a slowly varying medium, multiplicity of solutions for certain viscous flow problems, Stokes phenomena in the asymptotics of special functions, and the growth of instabilities.
Outline of Essay

The essay should take the form of a review. The first part should be a broad overview of asymptotics beyond all orders, but then one or two of the classes of problems to which ‘exponential asymptotics’ has been applied should be reviewed in detail, such as fluid mechanics (e.g. free surface flows, Saffman-Taylor fingering), hyperasymptotics, the regularity or otherwise of PDEs, ‘snaking’.

Relevant Course

Essential: Perturbation Methods.

References


75. Primordial Gravitational Waves from Cosmic Inflation .................

Dr A. D. Challinor

Cosmic inflation naturally provides a causal mechanism for generating all structure in the universe. Quantum fluctuations on microscopic scales were stretched outside the Hubble radius during inflation, providing the primordial perturbations that seeded large-scale cosmic structures and the temperature anisotropies in the cosmic microwave background (CMB). This same mechanism should also produce a primordial background of gravitational waves. If inflation occurred at high enough energies (around $10^{16}$ GeV) this background should be detectable in the CMB.

The basic predictions of simple inflation models have been verified to high accuracy with CMB temperature measurements (most notably from Planck). The one missing element is the detection of primordial gravitational waves. Detection of such a background would provide very
compelling evidence that cosmic inflation did occur, as well as determining the associated energy scale. Moreover, a detection would have significant implications for attempts to realise inflation in fundamental theory.

The best constraints on primordial gravitational waves come from observations of the $B$-mode polarization of the CMB, a specific curl-like pattern that is not excited by linear density perturbations. The measurements are very difficult since the amplitude is certain to be small and is now known to be subdominant to polarized emission from our Galaxy over nearly all of the sky. Although this primordial $B$-mode signal has not yet been detected, a new generation of multi-frequency (to allow removal of the Galactic emission) ground-based experiments are just beginning that should improve the current bound on the power in gravitational waves by around a factor of 10 (if they do not make the first detection). Plans are also being made for future satellite missions and a more ambitious ground-based experiment, both of which promise improvements by a further factor of 10.

Your essay should survey the theory behind the generation of primordial gravitational waves and their cosmological signatures. You may wish to address some of the following issues.

- What is the mechanism by which inflation can produce a background of gravitational waves, and how do gravitational waves leave imprints in the temperature and polarization of the CMB?
- Why is CMB polarization such a promising route for detecting this background (compared to, for example, the CMB temperature anisotropies)?
- What would we learn about the physics of inflation from a detection of primordial gravitational waves?
- Are there other cosmological sources of $B$-mode polarization and how can these be separated from the gravitational wave signal?
- What are the prospects for direct detection with, for example, space-based interferometers? How would such information complement that from CMB measurements?

**Relevant Courses**

*Essential:* Cosmology; Advanced Cosmology  
*Useful:* Quantum Field Theory

**References**

76. Ice Streams and the Rheology of Sub-Glacial Sediments
Dr J. A. Neufeld

Predictions of the rate at which glacial ice sheets transport land fast ice to the ocean remains an outstanding problem in climate studies with significant implications for the potential rate of sea level rise. In many settings, the majority of the ice is transferred in ice streams, which are highly localised regions of fast flowing ice. These ice streams are thought to arise through variations in the basal traction at the base of glaciers, often dominated by the dynamics of the sediment beneath the flowing ice stream or ‘sub-glacial till’. This project will review the scientific literature on possible mechanisms for localised ice streams, with particular attention to the mechanism and deformation of sub-glacial till. Suggested avenues to explore include (but are not limited to) (i) the extent to which the sub-glacial till may be modelled as a yield-stress fluid, and hence the depth to which the sub-glacial till may be mobilised, and (ii) the poroelastic response of subglacial till to variations in water pressure, and hence to the rheology of poroelastic materials. The coupling to models of glaciers as thin-viscous flows may be incorporated.

Relevant Courses

Useful: Slow viscous flows, Fluid dynamics of the solid Earth

References


77. Geological CO₂ Storage in Unbounded Reservoirs
Dr J. A. Neufeld

The geological storage of CO₂ is a primary means to reduce (or potentially reverse) the large anthropogenic emissions of a key greenhouse gas. When injected into porous and permeable rock, CO₂ is buoyant compared to ambient brine and therefore rises as a porous plume until it reaches an impermeable cap rock where it may continue spreading as a porous gravity current. This combination a porous and permeable reservoir overlain by an impermeable cap rock has been demonstrated in a number of pilot projects, but remains a limitation in selecting injection sites. Recently it has been suggested that processes that act to trap the CO₂, such as the dissolution of CO₂ into brine or the trapping of CO₂ within the pore space by surface tension, may be sufficient to limit the spreading of CO₂ even in reservoirs lacking a coherent cap rock. This essay will review the fluid mechanics of the buoyancy-driven flow of CO₂ in heterogeneous or fractured reservoirs. It should then discuss how trapping processes can then be incorporated into models of propagation with the aim of predicting the ultimate extent of CO₂ in unconfined reservoirs.
Relevant Courses

Slow viscous flows, Fluid Dynamics of the Solid Earth

References


78. Localised Convective Cells in Subcritical Convection

Professor M. R. E. Proctor

Many types of pattern forming instability can exhibit subcritical bifurcation, so that stable patterns can co-exist with a zero (unpatterned) state. Examples include convection in binary fluids and convection in an imposed magnetic field. If such bistable states exist there exist the possibility of fronts separating the zero from the patterned state. With two such fronts localised solutions are possible in which isolated groups of convection cells can exist in a large domain with no motion otherwise.

On one level these localised modes (if steady) can be thought of as homoclinic orbits of the complicated spatial equations describing steady convection, and in some cases can be looked at in terms of dynamical systems theory. Or they can be discussed as stable solutions of the time dependent convection equations. Some analysis and numerical work is possible revealing an intricate relationship between the many branches of localised modes with different numbers of convection cells. Most work has been done for two-dimensional solutions, though some interesting results for three-dimensional motion are starting to appear.

The essay should give an overview of the various types of such solutions in the literature, and then focus on detailed discussion of a particular aspect.

Relevant Courses

Essential: Part II Fluid dynamics or similar viscous fluids course
Useful: Part II Dynamical Systems or similar nonlinear dynamics course, Part III Stability and perturbation methods (Michaelmas term)

References

79. Advantages, Limitations and Challenges in Photoacoustic Imaging ......  
Dr O. Rath Spivack

Hybrid imaging techniques have attracted considerable attention in the past few years as a promising emerging medical imaging tool [1]. These techniques combine a high contrast modality with a high resolution modality, and have the potential for delivering the advantages of both techniques. Examples include ultrasound elastography, magnetic resonance elastography, photoacoustic tomography, magnetic resonance electric impedance tomography, and others. They are mostly not yet established as standard imaging techniques in a clinical setting, and are still the object of much theoretical investigation.

This essay should concentrate on photoacoustic tomography, describe the physical effect on which it is based, and explain the equations used to model the wave propagation and the inverse problem [2,3].

An overview should be given of the type of inverse problems that need solving, and of the range of validity of the models used [4,5]. Then, the essay could focus on the theoretical results for stability and uniqueness of solutions and the conditions needed [6], or on the validity of the approximations in applications, and perhaps the merits of different numerical methods for the solution [7,8]. The choice should depend on your interests and background.

Finally, the essay should give a view of current challenges and possible avenues for addressing them, especially in relation to the focus chosen in the main part of the essay.

Relevant Courses

Essential: Basic knowledge of PDEs, and particularly the wave equation, from any undergraduate course.

Useful: The Part III courses ‘Inverse problems in imaging’, or ‘Numerical Solutions of Differential Equations’, further background in PDEs, or in wave propagation in random media, or in numerical techniques such as Finite Element Methods may be useful, depending on the choice for the focus of this essay.

References

The Regularisation of Nonlinear Inverse Problems

Dr. O. Rath Spivack

Inverse problems arise whenever the cause of an observed effect is sought, and are of great importance in many applications, such as medical imaging, remote sensing, geophysical prospecting, non-destructive testing, and many more.

Many inverse problems in mathematical physics can be formally expressed as

$$Ax = y + \delta,$$  \hspace{1cm} (1)

where $A$ is an operator from a normed vector space $X$ into a normed vector space $Y$, $y \in Y$ is given data, often measured data, $\delta \in Y$ is a random error, and $x \in X$ is the unknown.

Usually such problems are ill-posed, and to ensure existence and uniqueness the solution is given in terms of the Moore-Penrose generalised inverse of $A$, denoted by $A^\dagger$, which solves the optimization problem of minimising the functional $\|Ax - y\|$. Nevertheless the problem often remains ill-posed because $A^\dagger$ is unbounded in most cases of interest, and the solution is unstable with respect to perturbations in the initial data. Therefore, additional regularisation techniques are needed in order to find meaningful solutions.

Such techniques have been extensively studied during the past 20 years or so and a fairly complete set of rigorous results exists in the case when $A$ is linear. The theory for nonlinear inverse problems is far less developed and fewer rigorous results exist.

This essay should explain the concept of regularisation, and the differences that arise in the regularisation of linear and nonlinear inverse problems.

According to your interests and background, it is possible to develop the essay in different directions. You might wish for example to focus on the theoretical results and proofs of convergence for the various methods, or on their respective merit in numerical implementations, or approach it from the point of view of finding the most appropriate regularisation for specific applications.

Relevant Courses

Useful: Some knowledge of functional analysis, or linear algebra and optimisation, from any course. The Part III course ‘Inverse problems in imaging’.

References

81. The Hidden Subgroup Problem and Kuperberg’s Algorithm

Dr M. Ozols

The Hidden Subgroup Problem (HSP) is an abstract computational problem that has been intensively studied in the context of quantum computing. One of the most celebrated results of quantum computing is Shor’s polynomial-time quantum algorithm for discrete logarithms and integer factorization, which also happens to solve a special instance of the general HSP. Following this discovery, polynomial-time quantum algorithms for many other instances of the general HSP have been found.

Nevertheless, for several instances of the general HSP polynomial-time quantum algorithm are still not known. One of them, the dihedral HSP, has a subexponential-time quantum algorithm due to Kuperberg [1, 2, 3]. This problem is important because of its connection to lattice problems in cryptography [4].

The hidden subgroup problem and its variations, such as the hidden shift problem, are currently a very active area of research. The HSP has been recently extended to continuous groups and a quantum algorithm for solving the HSP over $\mathbb{R}^n$ has been provided [5]. Variants of Kuperberg’s algorithm have recently found applications for constructing isogenies between elliptic curves [6] as well as for pattern matching [7].

The essay should introduce the general hidden subgroup problem, mention its most important special cases, and explain their relationships to well-known computational problems. The essay should then explain the standard method for attacking the general HSP. Optionally, the essay can also include a description of the weak Fourier sampling approach (this requires basic knowledge of representation theory) or a sketch of Kuperberg’s algorithm for the dihedral HSP.

A good starting point are the lecture notes by Andrew Childs [8] and the survey [9]. Basic introduction to the subject is available from the text-books [10, 11, 12]. Familiarity with quantum information and quantum computing (especially quantum algorithms) is necessary. Basic knowledge of representation theory and Fourier analysis would be useful.

Relevant Courses

Essential: Quantum Computation

References

Tensor network formalism—and the related notion of matrix product states—is a very useful tool for describing quantum mechanical systems. For example, this framework can be used to describe quantum circuits [3, 4], open quantum systems [2], as well as condensed matter systems [1].

Tensors are multi-linear generalizations of matrices since several indices are required to specify an entry of a tensor. A natural way of graphically depicting a tensor is by a node with several dangling edges, where each edge corresponds to one of the indices. Contraction of tensor indices is a generalization of matrix multiplication, and it is depicted by connecting the corresponding dangling edges. A tensor network is a collection of tensors, with some pairs of their indices contracted, and is hence represented by a graph.

The goal of this essay is to explore applications of the tensor network formalism in quantum computing. There are several alternative directions to pursue:

- classical simulation of quantum circuits [3, 4],
- quantum algorithms for evaluating tensor networks [5],
- addition of “virtual qubits” by means of classical post-processing [6].

The essay should introduce the notion of tensor networks, explain its relevance to quantum computing, and cover one of the applications mentioned above.

Relevant Courses

Essential: Quantum Computation

References


83. Detectability Lemma and its Applications in Quantum Hamiltonian Complexity .................................................. Dr M. Ozols

Quantum Hamiltonian complexity lies at the intersection of theoretical computer science and condensed matter physics. It studies local Hamiltonian problems that are quantum analogues of constraint satisfaction problems [1,4]. One of the most celebrated results in quantum Hamiltonian complexity is Kitaev’s proof of QMA completeness of the local Hamiltonian problem (also known as the “quantum Cook-Levin Theorem”) [1,4].

Detectability lemma is a simple but important tool in quantum Hamiltonian complexity and it helps to understand properties of system’s ground state [2,3]. Notably, detectability lemma can be used to obtain significantly simpler proofs of various fundamental results, such as exponential decay of correlations and 1D area law, that were originally proved using a much more complicated technique known as Lieb–Robinson bounds [5].

Based on reference [3], the essay should cover the proof of the detectability lemma and, optionally, also one of its applications: (i) exponential decay of correlations in the ground state of a frustration-free gapped Hamiltonian or (ii) 1D area law for frustration-free systems. For option (ii), a sketch of the proof is sufficient.

Relevant Courses

Essential: Quantum Computation
References


Dr J. B. Pitts

Geometry can seem to be a royal road to physics, especially in the case of General Relativity, in which one (pseudo-)Riemannian metric facilitates arriving at the field equations, defining causality, ascertaining the theory’s empirical consequences, and defining conserved quantities. The recent (re)consideration of massive gravity, bigravity, etc. shows renewed interest in multi-geometric possibilities of a sort evidently entertained already by Lobachevsky, Poincaré and Levi-Civita, but rarely considered on a sustained basis until recently, namely, having more than one example of a given geometric entity (2+ metrics, or 2+ volume elements, or 2+ connections, etc.) playing a role. In some cases it is difficult to decide whether a field is best regarded as geometry or matter.

In this more general context, what is space-time geometry and why does it matter?

Relevant Courses

*Useful* Space-time in Light of Particle Physics, General Relativity

References

Besides references from the course Space-time in Light of Particle Physics, see:


85. Computational Magnetohydrodynamics .......................... Dr N. Nikiforakis

The objective of this essay is to review the various forms of the equations describing unsteady, compressible magnetohydrodynamics (MHD) and the algorithms suitable for their numerical solution. The review of the equations should start from ideal MHD and then consider extensions including resistive, heating, viscous, radiative and multi-fluid (but not relativistic) effects. The review of the numerical algorithms should focus on upwind and centred, high-resolution, shock-capturing schemes, with particular focus on Riemann-problem-based methods. Two approaches for maintaining the exact divergence constraint of the magnetic field should be discussed. Finally, the student should provide numerical solutions of the equations in order to demonstrate some of the mathematical and/or numerical aspects discussed in the essay. The latter may be either solutions of the ideal MHD equations in one space dimension using a number of different numerical schemes (and comparisons against the exact solutions), or implementation of non-ideal extensions in 1D demonstrating the effects described in the essay, or multi-dimensional solutions of the equations and comparison against case-studies from the open literature.

Relevant Courses

Essential: Magnetohydrodynamics, Astrophysical Fluid Dynamics.

References


The Kochen-Specker theorem [1] gives tight constraints on how one could supplement orthodox quantum theory’s assignment of values to physical quantities. In foundations of quantum theory, it has had a large legacy. This essay considers just three strands. In approximate historical order, they are as follows.

(1): Bell’s own proof [2] of a version of the theorem motivated his famous non-locality proof [3], which of course led to countless studies of how locality constrained such suppletions of orthodox quantum theory. For Bell’s purpose in [3] was to ascertain whether any such suppletion, in order to replicate quantum mechanical statistics, must be non-local like the pilot-wave theory is. This went along with his advocacy [4] of the pilot-wave theory: which escapes the Kochen-Specker theorem, simply by not obeying its mathematical assumptions.

(2): Conway and Kochen [5] adapted some early 1980s theorems, which consider a spatially separated pair of spin 1 systems (not just one, as in the original Kochen-Specker theorem), to argue that these quantum mechanical systems showed a form of free will (!). This led to some rebuttals (unsurprisingly): but the recent analyses by Landsman et al. [6, 7] seem definitive.

(3): The Kochen-Specker theorem has various mathematical facets which have long been explored. For example, (A): one can ask what is the minimal number of quantities (specifically, projections) for which a proof can be given? And what about analogous theorems for multiple systems, or theorems where the argument depends on a choice of state? And (B): the theorem leads in to the general study of quantum contextuality, which can be cast in various general settings, for example topos theory (the initial paper being [8]). An up-to-date introduction to both (A) and (B), mostly via Mermin’s 1990 proof of the Kochen-Specker theorem, is [9].

The aim of the essay is to review these developments. Though all three strands could be pursued, it is probably wise to restrict yourself to combining two strands, the natural pairs being: (1) and (2); or (1) and (3).

### Relevant Courses

**Useful:** Philosophical aspects of quantum field theory

### References


Ever since the work of Newton and Wigner [1], it has seemed impossible to identify localized statevectors or position operators in Lorentz-invariant quantum theories that were not counterintuitive in some way: the most striking feature being superluminal propagation of the localized states. The topic turns partly on different schemes for associating operators with spacetime regions [2, 3, 4]. And it remains controversial. On the one hand, there are no-go theorems forbidding localization in certain senses, e.g. [5]. On the other hand, other authors argue that denying these theorems’ assumptions allows coherent, and even physically significant, notions of localization, e.g. [6,7]. The purpose of the essay will be to review these developments.

Relevant Courses

Essential: Quantum Field Theory, Philosophical Aspects of Quantum Field Theory

References

88. Two-Descent on the Jacobian of a hyperelliptic curve

Dr T. A. Fisher

The Mordell-Weil theorem states that the group of $K$-rational points $A(K)$ on an abelian variety $A$ defined over a number field $K$, is a finitely generated abelian group. Following the proof (a “descent calculation”) gives an upper bound for the rank of $A(K)$.

This essay should begin by describing the classical “number field” or “direct” method for 2-descent on elliptic curves $E/Q$. References for this include [1] and [2]. The essay should then explain how the method generalises to 2-descent on the Jacobian of a hyperelliptic curve: see [3], [4], [5], [6].

Relevant Courses

Essential: Elliptic Curves
Useful: Local Fields

References


89. Graph Ramsey Numbers

Professor I Leader

The graph Ramsey number $R(G,H)$ is the smallest $n$ such that whenever the edges of the complete graph on $n$ vertices are 2-coloured there is a red copy of $G$ or a blue copy of $H$. There has been a lot of exciting work recently, both on bounds for general graphs and also on bounds for specific graphs. For example, exact results have been obtained for the notorious problem of the Ramsey number of the cube versus a triangle, and good bounds have been found when we know about the maximum degrees of $G$ and $H$. The essay will look at some of this work.

Relevant Courses

Essential: None
Useful: Ramsey Theory
References


90. Graphs of Large Chromatic Number

Professor I Leader

What can we say about a graph with high chromatic number? Of course, it can contain a large complete graph. But what else? In other words, if we have a graph with high chromatic number but no large complete subgraph, what must it contain?

The essay will focus on recent work about some famous conjectures of Gyárfás, that concern the cycles in the graph. For example: if $G$ has high chromatic number but no triangle then it must contain a long induced cycle. These conjectures have resisted attack for some decades, but have now pretty much all been solved. The essay will look at some of these proofs.

Relevant Courses

Essential: A basic course in Graph Theory

Useful: None

References

[1] A. Scott and P. Seymour, Induced subgraphs of graphs with large chromatic number. I. Odd holes (Arxiv 1410.4118)

[2] M. Chudnovsky, A. Scott, and P. Seymour, Induced subgraphs of graphs with large chromatic number. III. Long holes (Arxiv 1506.02232)


91. Entropy production in active matter and reaction-diffusion systems

Professor M. Cates

Entropy production is a useful quantifier in determining the non-equilibrium character of systems far from equilibrium. Using fluctuation theorems, it is possible to compute the entropy production in a non-equilibrium system as it evolves through time by considering the ratio of the probability of the process running forward to the probability of it happening in reverse. Some current research has been devoted to studying this production in coarse-grained models of active matter [1].

However, in such models, the system is a priori known to be permanently outside of equilibrium as the underlying dynamics itself break time-reversal symmetry. It is therefore interesting to
compare this with what happens in reaction-diffusion systems which typically exhibit a slow transient towards equilibrium in which time-reversal symmetry is finally restored. In some cases, e.g. of type $A + B \rightarrow \emptyset$, this does, however, come at a price: due to the irreversibility of pair annihilation the entropy production is formally infinite at intermediate times.

The purpose of the essay will therefore be to compare and contrast current progress in determining the entropy production in active matter and reaction-diffusion systems. A successful essay is expected to review the relevant material, with suggested literature starting points being [1], [2] and [3]. An excellent essay might also consider how the existing methods need to be modified in more complex situations, for instance the reaction-diffusion system of the type described in [4], whose dynamics may need to be regularized - possibly by considering a small back-reaction rate.

**Relevant Courses**

*Essential: Theoretical Physics of Soft Condensed Matter, Statistical Field Theory*

*Useful: Biological Physics and Complex Fluids, Stochastic Calculus and Applications*

**References**


92. Oscillatory signals in the CMB power spectrum .................

Dr J. Fergusson and Professor P. Shellard

Standard models of inflation predict a scale-invariant spectrum of density perturbations with a slight red tilt. This leaves us with only a few parameters to constrain inflationary models, that is, the amplitude $A_s$, spectral index $n_s$ and tensor-to-scalar ratio $r$. Looking for deviations from this scale-invariant spectrum can offer further insight into the physics driving inflation, potentially opening a new observational window on fundamental physics. One particularly interesting possibility is an oscillatory scale-dependence in the primordial spectra (for a review, see [1]), this includes distinct signatures in the power spectrum and at higher order, such as the bispectrum and the trispectrum. These oscillations can have many causes from sharp features in the primordial potential or in the sound speed [2] through to interactions with fields other than the inflaton [3]. In axion monodromy scenarios, inspired by string theory [4], the potential is inherently periodic yielding a resonant or logarithmic oscillatory signal [5].

The goal of this essay will be to choose one of these mechanisms, to describe its motivation and derive the oscillations it produces in the power spectrum, and then to outline and interpret
any constraints available from recent observational data. A useful starting point, also with a theoretical overview, is the recent Planck paper "Constraints on inflation" [6] which details the most popular classes of models and provides further references for each (see the discussions from page 31 and page 36 onwards). Beyond the power spectrum, higher-order correlation functions offer further correlated information about oscillatory models, on which you might choose to comment (e.g. ref. [7]).

**Relevant Courses**

*Essential:* Cosmology

*Useful:* Advanced cosmology QFT GR

**References**

The following references offer examples of oscillatory-type inflation models yielding signatures that can be observationally constrained:


93. Homotopy Type Theory ........................................

*Prof. P.T. Johnstone*

Those who work with a type-theoretical (as opposed to set-theoretical) foundation for mathematics, such as category-theorists and theoretical computer scientists, have been aware for some time that, if types are viewed intensionally, they have the categorical structure of higher-dimensional groupoids. Such groupoids also arise in algebraic topology, as algebraic models of homotopy types. More recently, V. Voevodsky’s introduction of the ‘univalence axiom’ in type theory has opened up new and much more formal connections between type theory and homotopy theory, which hold out the hope that it may soon be possible to use foundational methods from type theory to prove new results in homotopy theory. The recent publication of the first book-form account of this material [1] brings it within the scope of a Part III essay.

**Relevant Courses**

*Essential:* Category Theory

*Useful:* Algebraic Topology
Inflation is the leading paradigm for explaining the origin of structure in the universe, but the microphysical cause of inflation remains elusive. Inflation probed energies that may be well beyond what can ever be reached by terrestrial particle physics experiments. At such high energies, the relevant field content and interactions are not known. Precision observations of the Cosmic Microwave Background (CMB) are consistent with inflation being driven by a single slowly rolling scalar field, but inflation may also have been caused by the collective motion of several fields.

This essay will explore how models of inflation involving two fields differ from single-field models. The essay should contain a careful discussion of the inflationary slow-roll conditions in two-field models and a derivation of the equations governing the perturbations. The decomposition of the perturbations into an ‘adiabatic’ mode and an ‘entropic’ (or isocurvature) mode should be discussed, and the possibility of (and conditions for) non-conservation of the curvature perturbation, \( \zeta \), on super-horizon scales should be demonstrated.

The essay should also contain a discussion of how two-field models can be observationally distinguished from single-field models. The essay may, but is not required to, contain a discussion of either: the observational feasibility of distinguishing single-field models of inflation from two-field models with the next generation of experiments; explicit analytic computations of the most relevant observables in a simple but non-trivial model of choice; numerical examples of the generated power spectrum and cross-correlations of the curvature and isocurvature perturbations, including their superhorizon evolution.

Relevant Courses

**Essential:** Quantum Field Theory, Cosmology.

**Useful:** Advanced Cosmology.

References

95. Applications of Topos Theory to Musical Analysis

Prof. P.T. Johnstone

Fifteen years ago Guerino Mazzola [1] proposed a programme of ‘mathematizing’ musical analysis on the basis of topos theory. Although anything like a complete overview of this programme would be well beyond the scope of a Part III essay, there are more specific applications of topos-theoretic ideas in music, such as Thomas Noll’s analysis of triads [2,3], which could serve as the basis for an essay which would first develop the mathematical tools required (using [4], for example), and then indicate how these can be used to enhance our understanding of a particular aspect of music. Anyone interested is advised to consult Professor Johnstone.

Relevant Courses

Essential: Category Theory

References


96. Recovery Guaranties in Compressed Sensing

Dr A. C. Hansen

In 2004, a new approach to signal processing and inverse problems called Compressed Sensing (CS) was launched by E. Candés, D. Donoho and T. Tao. Since the early developments the field has exploded and CS is now a mainstay in modern mathematical signal and image processing. It has had substantial impact in applications causing for example producers of Magnetic Resonance Imaging (MRI) machines for medical imaging to rethink many of the standard ways of doing sampling and reconstruction.

A quick description of CS is as follows. One is faced with the problem of solving a linear system $Ax = y$ where the matrix $A$ has fewer rows than columns, and hence can never be invertible. It is of course impossible to recover $x$ from $y$ in general under such conditions. However, in many cases, reasonable assumptions on $A$ and $x$ allow for recovery when using certain convex optimisation techniques. The problem is to understand how many samples $m$ (the dimension of $y$) is needed to recover $x$ given its dimension $n$ and assumptions on $x$ and $A$. The mathematical theory of CS is reasonably well understood in certain cases, although there are several open problems. This project is aimed at going into the details of some of the known proof techniques that have become popular in the last years. Some of the approaches are rather tricky, and as a result some rather serious mistakes have even found their ways into textbooks. The project will be about understanding the proof techniques and potentially provide better (and correct) solutions than the ones existing in the literature.
The candidate choosing this essay should have a strong background in analysis. Some knowledge of probability will be helpful as well as familiarity with convex optimisation.

**Relevant Courses**

*Essential:* Part II: Linear Analysis

*Useful:* Part III: Advanced Probability, Topics in Convex Optimisation, Topics in Mathematics of Information.

**References**


97. **Sparse recovery techniques in infinite-dimensional inverse problems . . . .**

Dr A. C. Hansen

Infinite-dimensional inverse problems, meaning problems of the form $Tx = y$ where $x$ is in some infinite-dimensional space, occur everywhere in the sciences ranging from medical imaging via microscopy to astronomy and surface scattering. The reason why we typically end up with infinite-dimensional inverse problems is that the operator $T$ describes a transform that is dictated by the physics of the apparatus that produces the vector $y$. This can, for example, be the Fourier or the Radon transform, as in surface scattering or Computerised Tomography (CT), or the Walsh transform that occurs in fluorescence microscopy.

Recently, the use of sparse recovery techniques has had a great impact on contemporary inverse problems. In particular, by utilising the fact that many problems have an underlying sparse structure in for example a wavelet representation, one may outperform many of the state-of-the-art techniques that were built on the philosophy of inverting $T$. The project is about analysing recent techniques based on optimisation problems using X-lets (wavelets is in this big family) and sparse structures. The key here is really the infinite-dimensional approach. Note that even though, at the final computational stage, one must have a finite problem, the way to this discretisation is paved with difficulties. In particular, naive discretisations may lead to highly suboptimal methods.

The candidate choosing this essay should have a strong background in functional analysis. Some knowledge of inverse problems will be helpful as well as some familiarity with optimisation.
Relevant Courses

Essential: Part II: Linear Analysis, Part III: Functional Analysis
Useful: Part III: Inverse Problems in Imaging, Topics in Convex Optimisation, Topics in Mathematics of Information.

References


98. The explicit isogeny problem for elliptic curves ...............................  
Dr T. A. Fisher

An isogeny of elliptic curves is a map that is both a (non-constant) morphism of algebraic curves and a group homomorphism. Given elliptic curves $E_1$ and $E_2$ defined over a finite field $k$, and a positive integer $r$, the explicit isogeny problem asks if there is an isogeny $\psi : E_1 \to E_2$ of degree $r$, and to find rational functions defining such an isogeny if there is one.

Methods for solving this problem that involve computing the modular polynomial $\Phi_r$, necessarily have complexity at least cubic in $r$. This essay should describe the quadratic time method in [2], improving on an earlier method of Couveignes that was only practical over finite fields of small characteristic. The essay should include some discussion of the earlier methods, together with background material on endomorphism rings of elliptic curves, Velu’s formulae, modular polynomials, and isogeny volcanoes (see [3] and [4]). Ideally the essay should be illustrated by the candidate’s own numerical examples.

Relevant Courses

Essential: Elliptic Curves
Useful: Local Fields
References


99. Min-max solutions to the Allen-Cahn equation and the existence of minimal hypersurfaces. .................................................................

Professor N. Wickramasekera

Given a compact Riemannian manifold $M$ and an integer $k$ with $1 \leq k \leq \dim M$, it is a classical problem whether there is an embedded $k$-dimensional closed minimal surface $\Sigma \subset M$. Since such a surface is a critical point of the $k$-dimensional area functional, this problem can be approached variationally, by directly constructing a critical point of the area. In the case $k = 1$, the problem is that of existence of closed geodesics and has been extensively studied. When $k = \dim M - 1$ (i.e. the hypersurface case), the combined work of Almgren, Pitts, Schoen-Simon dating back to the early 1980s implemented this variational approach, producing a minimal hypersurface (with possibly a small singular set of codimension 7) that is a min-max critical point of the hypersurface area.

Recent work of several authors has lead to a fundamentally different approach to the existence of minimal hypersurfaces. This approach is more PDE theoretic, and is based on a min-max construction, for each small $\epsilon > 0$, of a suitable real valued solution $u_\epsilon$ to the (elliptic) Allen-Cahn equation $\epsilon \Delta u - \epsilon^{-1} W(u) = 0$ on $M$, where $W$ is a fixed smooth double-well potential satisfying appropriate conditions (e.g. $W(u) = 1/4(1-u^2)^2$). The minimal hypersurface arises as a weak limit of the nodal sets of $u_{\epsilon_j}$ for a sequence $\epsilon_j \to 0^+$.

The essay should start by explaining the main geometric and analytic ideas behind this new proof following [2]. It should then proceed to explore the PDE aspects in depth, giving a detailed account of the PDE min-max tools upon which the proof in [2] is built, by following for example [1].

Relevant courses: Essential: Elliptic PDEs
Useful: Differential Geometry


100. Moduli space of curves ..........................................................

Professor M. Gross

Points of a moduli space correspond to geometric objects (or their isomorphism classes) of some fixed kind. The prototypical example is the projective space, seen as the collection of
lines passing through the origin, or more generally, the Grassmanian, collection of all linear subspaces of given dimension of some vector space. Moduli spaces also themselves have some geometric structure, for example of a variety or a scheme, and studying their geometry can bring insight into the objects that the moduli space classifies. The aim of this essay is to review the construction and some properties of the moduli space $M_g$ of all algebraic curves of given genus, and of its compactification to the moduli space of stable curves $\overline{M}_g$. These spaces cannot be realised as varieties or even schemes, but are rather examples of much more general objects called (Deligne-Mumford) stacks, which were partly introduced for tackling various moduli problems. Part of the essay will be devoted to discussing stacks in general.

The original reference for the subject is [1], while [2] gives a slightly different view and collects some other results. For a general survey of stacks, see [3].

Relevant Courses

**Essential:** Part III Algebraic Geometry

**Useful:** Topics in Algebraic Geometry

References


101. **SQ-universality and the word problem in groups**

Dr M. Chiodo

A countable group is *SQ-universal* if every countable group embeds in one of its quotient groups (here S stands for Subgroup, and Q for Quotient). Higman, Neumann and Neumann [1] proved that the free group on 2 generators, $F_2$, is SQ-universal.

Classical results in group theory show that there are finitely presented groups for which the *word problem*, of deciding whether or not a word represents the trivial element, is algorithmically incomputable. There are several important constructions and embedding results related to this, many of which show that one can simulate the action of a Turing machine within a finitely presented group. These are surveyed in [4].

Miller [3] showed that there is a SQ-universal group $G$ such that the word problem is unsolvable in every non-trivial quotient of $G$ (and of Turing degree 0' when the quotient is recursively presented). He makes use of *recursively inseparable sets* of integers; disjoint sets which cannot be distinguished by any recursive function. He also relies heavily on the Adian-Rabin construction for finitely presented groups, as well as Higman’s Embedding Theorem.

Houcine [2] has recently extended the work of Miller, to show that for every recursively enumerable existential consistent theory $\Gamma$, there exists a finitely presented SQ-universal group $H$ such that $\Gamma$ is satisfied in every nontrivial quotient of $H$. He also shows that a finitely generated group $G$ has solvable word problem if and only if there exists a finitely presented group $H$ such that $G$ is embeddable in every non-trivial quotient of $H$. This is a new and interesting algebraic characterisation of finitely generated groups with solvable word problem.

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The purpose of this essay is to give an overview of the results and techniques of Miller [3] and Houcine [2], and to present the work of Houcine [2] in a purely group-theoretic way without the notational dependence on model theory.

Relevant Courses

**Essential:** Part III Decision problems in group theory.

**Useful:** Part III Geometric group theory, Part II Automata and formal languages.

References


102. Lattice QCD calculation of weak matrix elements .................

Dr M B Wingate

Of the three fundamental forces described by the Standard Model of particle physics, the dynamics of the weak force are the least certain. Despite the discovery of the Higgs boson (or a Higgs-like scalar) in 2012, theorists still question whether the Standard Model description of electroweak symmetry breaking (EWSB) is complete. If there are any new particles or dynamics which play a role in EWSB, then they could alter the Standard Model description of quark flavour interactions, the Cabibbo-Kobayashi-Maskawa (CKM) mechanism.

Due to confinement, quark decays are not measured directly by experiments. only hadron decays. Accurate QCD calculations are required in order to connect quark-level fundamental parameters to hadronic measurements. Numerical calculations using lattice gauge theory allow determination of the necessary weak matrix elements.

The aim of essays submitted with this title will be to summarize the key ingredients and consequences of a few lattice QCD calculations relevant for weak hadronic decays. Each author should choose a few related hadronic decays to consider, e.g. decays of the $K$, $D$, or $B$ meson. Alternatively one could study neutral $K$ or $B$ meson mixing. Some examples will require more discussion of the weak physics than others. With an example in mind, the author should then introduce the relevant notions from lattice gauge theory.

The successful essay will:

1. Describe one or a few measurements to be studied in the essay. In some cases some substantial theoretical discussion will be required to connect the Standard Model process to the weak matrix element(s) which will be determined using lattice QCD.
2. Briefly introduce the key ideas in lattice QCD which will be relevant to the physical processes discussed. Some examples will warrant a discussion of lattice fermions and chirality, while others will require some mention of methods for heavy quarks on the lattice. It will be necessary to discuss the renormalization of the weak operator.

3. Review recent, unquenched calculations of the necessary matrix elements, including a discussion of the uncertainties.

Relevant Courses

*Essential: Quantum Field Theory & Advanced Quantum Field Theory*

*Useful: Standard Model &; to a lesser extent, Statistical Field Theory*

References


103. There is no abelian scheme over \( \mathbb{Z} \) .................................

Professor A. J. Scholl

A classical theorem of Tate from the 1960s states that there is no elliptic curve over the integers with discriminant ±1. This means that there does not exist an elliptic curve over the rationals which has good reduction at every prime. In the 80s, Abrashkin [1] and Fontaine [3] proved independently that the same statement held for abelian varieties over the rationals of arbitrary dimension, using the theory of finite flat group schemes over extensions of \( \mathbb{Z}_p \). Subsequently both authors have extended these results in different directions, to the study of smooth proper schemes over \( \mathbb{Z} \).

The aim of this essay is to give an account of some of these results and some of the underlying \( p \)-adic Hodge theory.

Relevant Courses

*Essential: Local Fields, Elliptic Curves, Algebraic Geometry.*

References

104. MCMC Methods for Functions: Modifying Old Algorithms to Make Them Faster .......................................................... Dr S. S. Singh

In [1] the authors aim to explore the possibilities of modifying existing Markov Chain Monte Carlo (MCMC) methods to make them usable in high-dimensional settings. They claim that many problems in applications result in the need to probe a probability distribution for functions, e.g. Bayesian nonparametric statistics or conditioned diffusion processes. Standard MCMC algorithms that are used to solve these problems suffer from the curse of dimensionality, as they become arbitrarily slow under mesh-refinement. Therefore modifications become necessary, aiming to make those algorithms faster and thus usable in high-dimensional settings. The authors describe a variety of such adaptions, which are possible whenever the posterior has a density with respect to some Gaussian process or Gaussian random field reference measure.

The student is expected to scrutinise the scope of problems that can be addressed, using ideas in [1] (and more generally ideas proposed by other authors as well), the efficacy of their MCMC modifications, work out the underlying principles, find new potential applications of these algorithms, evaluate their weaknesses, and maybe propose own modifications or formulate future research questions.

Relevant Courses

Useful:
Bayesian Modelling and Computation, Gaussian Process, and Advanced Probability

References


105. Discrete-state hidden Markov models for non Euclidean data ......... Dr D. Pigoli

Hidden Markov models (HMMs) are statistical models characterised by an underlying unobserved (hidden) process generating an observable sequence. In the discrete-state case, this underlying process travel between a finite (or countable) set of states. These models have been extremely useful for segmentation of time series, with the transition times of the hidden process marking change points in the observable sequence. In some cases, the observed sequence may take value in a space that is non Euclidean, for example data may be angles (wind directions) or shapes (protein conformations). This poses new challenges both in the statistical theory and in the practical implementation.

The idea of this essay is to review existing methods for HMMs with non Gaussian data ([1],[2]) and to consider their application to directional ([3]) and/or shapes ([4]) data and how this can be implemented in R. This can be complemented with simulation studies to show the performance of the method for sequence segmentation.
Relevant Courses

*Useful:* Topics in Statistical Theory, Applied Statistics

References


106. The Gaussian free field and Liouville quantum gravity

*Dr J. P. Miller*

The Gaussian free field (GFF) [1] is a two-time dimensional analog of the Brownian motion. Just as the Brownian motion describes the scaling limit of many types of random curves, the GFF describes the scaling limit of many types of random surface models. The GFF is also closely connected with many different objects in probability, including random walks and the Schramm-Loewner evolutions (SLE). The focus of this essay is on a certain type of random geometry that can be generated from the GFF, namely Liouville quantum gravity (LQG). LQG was introduced by a physicist named Polyakov in the 1980s as a model for a random surface. Polyakov’s theory has proved to be very powerful for deriving exponents associated with various probabilities for statistical physics models on planar lattices, such as percolation, the Ising model, and simple random walk. These derivations come via the so-called KPZ relation, a heuristic which serves to relate exponents for such models on a planar lattice with the corresponding exponent on a random surface. Beginning with the work of Duplantier and Sheffield [2], in recent years, there has been tremendous interest in the probability community to put the framework of LQG and the KPZ relation on firm mathematical ground.

A successful essay will give a review of the GFF [1], describe the construction of the Liouville quantum gravity measure [2], explain its connection to SLE [3], and discuss the KPZ relation.

Relevant Courses

Advanced probability, Schramm-Loewner evolutions, Stochastic Calculus.

References


107. Central limit theorems for empirical processes

Professor R. Nickl

Given the importance of the Central Limit Theorem (CLT) in traditional parametric statistics, it was natural that infinite-dimensional versions of the CLT would be of key importance to modern ‘high- and infinite dimensional’ statistical theory. The first version was proved (somewhat incorrectly) by M. Donsker in 1952 for distribution functions on $\mathbb{R}$, and a abstract theory emerged in the 70s and 80s driven by the work of R.M. Dudley, E. Giné, J. Zinn, D. Pollard, V. Koltchinskii and others, a theory whose techniques have been highly useful in mathematical statistics ever since. In abstract settings it is of interest to consider for which classes of functions $\mathcal{F}$ the central limit theorem holds for the empirical process

$$\sqrt{n}(P_n - P)f = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (f(X_i) - \mathbb{E}f(X)), \quad f \in \mathcal{F},$$

in the space $L^\infty(\mathcal{F})$ of bounded real functions defined on $\mathcal{F}$. Such classes $\mathcal{F}$ are called ‘Donsker’ class of functions. A related family of classes $\mathcal{F}$ are those that are ‘pregaussian’, providing the existence of a sample-continuous Gaussian limit process. In particular if a class of functions is P-Donsker then it is also P-pregaussian, however the converse does not hold unless further constraints are imposed upon the class. An interesting question is what precise conditions have to be added to ‘pregaussian’ in order to obtain the Donsker property, and such theorems are quite useful in nonparametric statistics, including amongst others recent applications to inference for Lévy measures ([2], [3], [4]).

The main aims of this essay are to first give an overview of how the central limit theorem for empirical processes works in an infinite dimensional setting, and subsequently explore the nature of the relationship between P-pregaussian and P-Donsker classes of functions, as well as discuss how these techniques can be generalised in a way relevant in applications to nonparametric statistics.

Relevant Courses

Some basic background in Probability Theory and Functional Analysis will be helpful, and the Part III course Gaussian Processes is recommended.

References

108. Exploring the statistical and biological utility of deep learning

Dr S. J. Eglen

Neural networks have been studied extensively since the 1970s, both for studying brain function and for purely statistical classification. Their popularity waned in the early 1990s as many of their limitations were discovered. However, there has recently been a resurgence in ‘deep learning’: neural networks using a large number of hidden layers [1]. This essay will review what deep learning is, and compare their abilities with other modern statistical tools for classification. A second component of the essay could be to review whether deep learning models tell us anything new about brain function [2] and if they can be used to understand unsupervised forms of learning in the brain [3]. Finally, a practical element of the essay could involve implementing these networks and running novel comparisons [4].

Relevant Courses

Essential: Computational Neuroscience

Useful:

References


109. Around the Nash-Moser theorem

Professor C. G. A. Mouhot

Nash-Moser theorem is an analogy of the inverse function theorem. It originated from Nash’s revolutionary solution to the isometric embedding problem. It was later developed by Moser into a powerful iterative scheme technique for constructing solutions to PDEs in spite of the phenomenon of “loss of derivatives” in the estimates. Moser also used this iterative technique in his proof of the KAM theorem with finite regularity. Most theorems in PDEs established first with the help of a Nash-Moser scheme have later been reproved with simpler fixed-point arguments, with the notable exception of the KAM theorem.

The essay should include (compulsory) (1) a review of the basics of pseudo-differential calculus and Littlewood-Paley decomposition following [1,2], (2) a detailed presentation of the Nash-Moser theorem and its application to the embedding problem following [1,3], and (desirable) (3) the study of the short alternative proof of the embedding theorem by G"unther [4] that managed to get rid of the Nash-Moser scheme.

Relevant Courses

Essential: Undergrad courses in basic analysis

Useful: Analysis of Partial Differential Equations, Functional Analysis
This essay is a description of the nature of the BMS group and its applications to the physics of gravitational radiation, gravitational memory and the physics of black holes. The starting point is the production of gravitational radiation and so some methods of producing gravitational radiation should be discussed together with the quadrupole formula as an approximation to the flux of radiation. There should also be a brief discussion of the recent observations at LIGO. Then there should be a detailed description of the nature of asymptotically flat spacetimes including the form of the metric near null infinity. The BMS group should be described in terms of its asymptotic symmetries. Then there should be a description of the energy flux formula and its relation to BMS transformations. This could be extended to include a discussion of the radiation of angular momentum. Finally, the treatment should be extended to include BMS transformation on black hole horizons. The treatment of gravitational symmetries is paralleled by similar phenomena in electrodynamics and as a prelude to the gravitational case, this could also be discussed.

Relevant Courses

Essential: General Relativity, Black Holes.
Useful: Quantum Field Theory.

References

111. Free molecular flow and boundary effects

Professor C. G. A. Mouhot

The purpose of this essay is to understand several recent works devoted to the rigorous mathematical study of free molecular flows in a vessel with diffusive boundary conditions. This is studied in the kinetic regime in terms of a PDE evolution problem, i.e. the free transport equation (collisionless Boltzmann equation). The essay should include (compulsory) (1) a short review of the basics of kinetic theory in chapters 1, 3 and 4 of [1], (2) a detailed presentation of the asymptotic convergence results in [2,3,4] with their proofs, and (desirable) (3) how it contrasts with the case of collisional gases studied in [5].

Relevant Courses

Essential: Undergrad courses in basic analysis
Useful: Analysis of Partial Differential Equations, Functional Analysis

References