Formalising mathematics in Isabelle

Fundamental lemma of calculus of variations
Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be continuous and $\forall h : [a, b] \subset \mathbb{R} \to \mathbb{R}$ continuous with $h(a) = h(b) = 0$ we have $\int_a^b f(x)h(x)\mathrm{d}x = 0$.
Then $\forall x \in [a, b]$, $f(x) = 0$.

Figure 1: The fundamental lemma of calculus of variations

About Isabelle

Isabelle, developed in big part by Professor Lawrence Paulson (Computer Lab), is an Interactive Theorem Prover (ITP) in Higher Order Logic. The goal of an ITP is to provide an environment for machine assisted proofs. The user is responsible for writing the proof in the appropriate language and syntax whilst Isabelle checks the logic of every argument. A complete proof in Isabelle is therefore fully verified error (bug) free. There is an additional tool, Sledgehammer, which is an automatic theorem prover which is able to prove simple statements that form part of a proof.
Since Mathematics is not a science that can be verified through experiment, being able to ascertain the proof of a theorem is extremely attractive and one might expect it's use to grow in the future.
Isabelle is not only used in Mathematics, it has many applications in the verification of security protocols, properties of programs and many other objects of interest in Computer Science. The archive of formal proofs contains all such results, it can be accessed at https://www.isa-afp.org.

Using Isabelle to formalise some basics of the calculus of variations

The aim of the project was set on formalising some basic results in the calculus of variations. For the most part this involves proving various lemma about continuous functions and their integrals.
Isabelle already contains an extensive library for Multivariate Analysis which includes formalisations for derivatives and gauge integration. There is also the Lebesgue integral defined in the Probability formalisation. These provide all the necessary definitions and basic properties. Nonetheless there is still a vast amount of mathematical knowledge to add, proving a lemma often means going back to prove some needed facts. In particular having two different formalisations of integration means that some facts are available for one and not the other. Switching back and forth between both can be a challenge. An effort in this project has been made to prove everything for both forms of integration.
An interesting aspect of the project is the interplay between being able to prove the mathematics and being able to use Isabelle to do so. It is no doubt much harder to do the latter as becoming proficient with the different syntax and definitions of each previous formalisation used is an exercise in itself.

Calculus of variations

Calculus of variations is a branch of mathematical analysis which concerns itself with functionals which map functions to real numbers.
This plays an important role in physics where variational methods are used to obtain equations of motion. Consider for example the action functional: $S[x] = \int L(x(t), \dot{x}(t), t)\mathrm{d}x$ where the Lagrangian L is a function of position x, velocity $\dot{x}$ and time t. Finding the extrema of S leads to the Euler-Lagrange set of differential equations. Using these, one can derive Noether's Theorem, one of the most important results in theoretical physics, which links symmetries to conserved quantities.
Variational methods find a use in many areas such as Bayesian inference and machine learning amongst others.

```
lemma fundamental_calculus_of_variation:
fixes f::"real ⇒ real"
fixes h::"real ⇒ real"
assumes "a < b"
assumes "continuous_on {a .. b} f"
assumes "∀h. continuous_on {a..b} h ∧ h a = 0 ∧ h b = 0 ⟶ LBINT x=a..b. f x * h x = 0"
shows "∀x∈{a .. b}. f x = 0"
proof (rule ccontr)
assume " ¬ (∀x∈{a .. b}. f x = 0)"
def r ≡ "(%x. -(a-x)*(b-x))"
(* some basic properties of r *)
have "r a = 0" [1 lines]
have "r b = 0" [1 lines]
have "∀x∈{a .. b}. r x ≥ 0" [1 lines]
have "∀x∈box a b. r x > 0" [1 lines]
have "continuous_on {a .. b} r"  [8 lines]
(* basic properties of the integral of f^2*r *)
have "continuous_on {a .. b} (%x. (f x)^2 * r x)" [9 lines]
then have "interval_lebesgue_integrable lborel a b (%x. (f x)^2 * r x)" [1 lines]
moreover have "∀x∈{a .. b}. (f x)^2 * r x ≥ 0" [1 lines]
then have "LBINT x=a..b. (f x)^2 *r x ≥ 0" [2 lines]
(* if there is some c such that f is non zero then the integral is strictly positive *)
have "∃c∈box a b. f c ≠ 0" sorry
obtain c where "c∈box a b" "f c ≠ 0" [2 lines]
then have "LBINT x=a..b. (f x)^2 * r x >0" [20 lines]
(* now pick h such that the integral is zero by assumption, giving a contradiction *)
def h ≡ "(%x. f x * r x)"
have "LBINT x=a..b. f x * h x = 0" [11 lines]
then have "LBINT x=a..b. (f x)^2 *r x = 0" [10 lines]
then show False [2 lines]
qed
```

Figure 2: Proof by contradiction of the fundamental lemma of calculus of variations in Isabelle.

About the Author

Alexander Hicks is a recent Master of Advanced Study (Part III) in Mathematics graduate, with a focus on mathematical physics.
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