Define a group $G$ as
\[ G = \{ g : g = (g_1, g_2, g_3, g_4), g_i \in \{0, 1 \} \} \]
with the group operation being the componentwise addition modulo 2, that is
\[ g^i g^j = g^i g^j = (g_1^i + g_1^j, g_2^i + g_2^j, g_3^i + g_3^j, g_4^i + g_4^j) \text{ (mod 2)} \]
(2)
For each $g \in G$, define a transformation $d(g)$ on the quark field and the antiquark field as
\[ d(g)(\psi(x)) = e^{ix \cdot \pi_{g} \xi} M_{g} \psi(x) \]
\[ d(g)(\bar{\psi}(x)) = e^{ix \cdot \pi_{g} \xi} \bar{\psi}(x) M_{g}^{\dagger} \]
(3)
and its negative counterpart $-d(g)$ as
\[ -d(g)(\psi(x)) = -e^{ix \cdot \pi_{g} \xi} M_{g} \psi(x) \]
\[ -d(g)(\bar{\psi}(x)) = -e^{ix \cdot \pi_{g} \xi} \bar{\psi}(x) M_{g}^{\dagger} \]
(4)
whereby $\pi_{g}$ are the 16 corners of the Brillouin zone
\[ \pi_{g} = \frac{\pi}{a} \]
(5)
and $M_{g}$ are the matrices defined as
\[ M_{g} = \prod_{\mu, g_{\mu} = 1} M_{\mu} \]
(6)
with
\[ M_{\mu} = i \gamma_{5} \gamma_{\mu} \]
(7)
The naive action for free fermions on the lattice given by
\[ S_{0}(\psi) = a^{4} \sum_{x} \left( \sum_{\mu} \bar{\psi}(x) \gamma_{\mu} \frac{1}{2a} \left[ \psi(x + a \hat{\mu}) - \psi(x - a \hat{\mu}) \right] + m \bar{\psi}(x) \psi(x) \right) \]
(8)
is invariant under this set of 32 discrete transformations. We note that these transformations compose with one another according to the following
\[ d(g') \circ d(g')(\psi(x)) = e^{ix \cdot (\pi_{g} + \pi_{g'})} M_{g'} M_{g} \psi(x) = \varsigma_{ij} e^{ix \cdot \pi_{g} \xi} M_{g'} \psi(x) \]
\[ d(g') \circ d(g')(\bar{\psi}(x)) = e^{ix \cdot (\pi_{g} + \pi_{g'})} \bar{\psi}(x) M_{g'}^{\dagger} M_{g}^{\dagger} = \varsigma_{ij} e^{ix \cdot \pi_{g} \xi} \bar{\psi}(x) M_{g'}^{\dagger} \]
(9)
where $\varsigma_{ij}$ are such that
\[ M_{g'} M_{g} = \varsigma_{ij} M_{g'} M_{g} \]
(10)
We see that the 32 transformations given in (3) and (4) form a group, the “doubling symmetry” group $D$, with its structure inherited from the group $G$ such that
\[ D = \{ \pm d(g) : d(g')d(g') = \varsigma_{ij}d(g'g'), g \in G \} \]
(11)
In other words, we have
\[ q : D \rightarrow D/\{ \pm I_{D} \} \cong G \]
(12)
We are interested in finding irreducible representations of the doubling symmetry group $D$. To proceed, we would first like to look at the irreps of group $G$, which can then be lifted up to irreps of $D$ by composing with the quotient map $q$ from (12). To determine the irreps of $G$, we make use of its following properties
1. $G$ is an abelian group of order 16
2. all group elements of $G$, except the identity, have order 2
3. $G$ is generated by its 4 elements $g^{1} = (1, 0, 0, 0), g^{2} = (0, 1, 0, 0), g^{3} = (0, 0, 1, 0), g^{4} = (0, 0, 0, 1)$
By property (1), $G$ has 16 inequivalent 1-dimensional irreps. By property (2) such an irrep can only go to itself or be multiplied by a minus sign under any group element of $G$. By property (3) each irrep of $G$ is uniquely determined by how it transforms under $g^{1}, g^{2}, g^{3}, g^{4}$. Therefore we label the 16 irreps of $G$, $\rho_{G}(\xi)$, by a 4-component vector
\[ \xi = (\xi_{1}, \xi_{2}, \xi_{3}, \xi_{4}), \quad \xi_{\mu} \in \{ \pm 1 \} \]
(13)
such that the corresponding vector space $\langle v(\xi) \rangle$ transforms under $G$ according to
\[ \rho_{G}(\xi)(g^{\mu})(v(\xi)) = \xi_{\mu} v(\xi), \quad \mu \in \{ 1, 2, 3, 4 \} \]
(14)
Lifting up, we get 16 1-dimensional irreps of $D$, $\rho_D^1(\xi)$ on the vector space $\langle v(\xi) \rangle$, such that

$$\rho_D^1(\xi)(\pm d(g)) : v(\xi) \mapsto v(\xi) \prod_{\mu, g_\mu = 1} \xi_\mu \quad (15)$$

Define a matrix group $M$ as

$$M = \{ \pm M_g : g \in G \} \quad (16)$$

with the group operation being the usual matrix multiplication, we have

$$D \cong M \quad (17)$$

The doubling symmetry group $D$ breaks into 17 conjugacy classes

$$\{ \pm d(g) \}_g \in G \setminus \{ I_D \} \bigcup \{- I_D \} \bigcup \{ I_D \} \quad (18)$$

and as for a finite group number of irreps equals number of conjugacy classes, we deduce that there is a last 4-dimensional irrep of $D$, denoted by $\rho_D^4$, such that

$$\rho_D^4(\pm d(g)) = \pm M_g \quad (19)$$

and obtain the full character table of the doubling symmetry group $D$ as follows

\[
\begin{array}{ccc}
\uparrow & \{ \pm d(g) \}_{g \in G \setminus \{ I_D \}} & \rightarrow & \{ -I_D \} & \{ I_D \} \\
\rho_D^1 & \prod_{\mu, g_\mu = 1} \xi_\mu & 1 & 1 \\
\rho_D^4 & \leftarrow 0 & \rightarrow & -4 & 4 \\
\end{array}
\]

where the first row lists the 17 conjugacy classes of $D$, and the first column lists its 17 irreps.

We return to the representation of $D$ on the quark field. This representation has no overlap with any of the irreps $\rho_D^1(\xi)$, because the identification of $-I_D$ to $I_D$ in the 1-dimensional irreps is unphysical. Therefore the representation of $D$ on the quark field reduces to copies of irrep $\rho_D^4$, and by the same reasoning so does the representation of $D$ on the antiquark field.

To analyse the diquark representation and the antiquark-quark representation, it suffices to look at $\rho_D^4 \otimes \rho_D^4$. Recalling that the character of a tensor product representation is the product of the characters of its factors, we compute the character of $\rho_D^4 \otimes \rho_D^4$ as

\[
\begin{array}{ccc}
\uparrow & \{ \pm d(g) \}_{g \in G \setminus \{ I_D \}} & \rightarrow & \{ -I_D \} & \{ I_D \} \\
\rho_D^4 \otimes \rho_D^4 & \leftarrow 0 & \rightarrow & 16 & 16 \\
\end{array}
\]

Employing the projection formula for the multiplicity of irrep $\rho_D^1(\xi)$ in the representation $\rho_D^4 \otimes \rho_D^4$

$$m_{\rho_D^1(\xi)}^{\rho_D^4 \otimes \rho_D^4} = \langle \chi_{\rho_D^1(\xi)}, \chi_{\rho_D^4 \otimes \rho_D^4} \rangle \quad (20)$$

whereby the inner product $\langle \ , \ \rangle$ on the characters of any two representations, $\alpha$ and $\beta$, for a general finite group $\Omega$ is defined as

$$\langle \chi_\alpha, \chi_\beta \rangle = \frac{1}{|\Omega|} \sum_{\omega \in \Omega} \chi_\alpha(\omega) \chi_\beta(\omega) \quad (21)$$

we decompose the tensor product representation into

$$\rho_D^4 \otimes \rho_D^4 = \bigoplus_{\xi} \rho_D^1(\xi) \quad (22)$$

i.e. the diquark representation and the antiquark-quark representation are both described by the 16 1-dimensional irreps of the doubling symmetry group.
We look at meson operators of the form
\[ \tilde{\psi}(x) \Gamma^m \psi(x + a\hat{\mu}_1 + \cdots + a\hat{\mu}_r), \quad 0 \leq r \leq 4 \]
whereby \( \hat{\mu}_1, \ldots, \hat{\mu}_r \) are distinct links, i.e. space-time vectors of unit length along directions given by the indices \( \mu_1, \ldots, \mu_r \) respectively. Making use of the identity
\[ M_g^\dagger \gamma_{u_1} \cdots \gamma_{u_k} M_g = (-1)^{g_{u_1} + \cdots + g_{u_k}} \gamma_{u_1} \cdots \gamma_{u_k}, \quad \mu_1, \ldots, \mu_k \in \{1, 2, 3, 4\} \]
we establish, for each choice of links, a one-to-one correspondence between the gamma matrices \( \Gamma_{\mu_1 \cdots \mu_r}^m \) and the irreps \( \rho_D^1(\xi) \) of the antiquark-quark representation
\[ \Gamma_{\mu_1 \cdots \mu_r}^m(\xi) = \gamma_{u_1} \cdots \gamma_{u_r} \prod_{\mu, \xi, = -1} \gamma_u \ \Leftrightarrow \ \rho_D^1(\xi) \] (25)

Explicitly, we have the above relation as
\[ \pm d(g) : \tilde{\psi}(x) \Gamma_{\mu_1 \cdots \mu_r}^m(\xi) \psi(x + a\hat{\mu}_1 + \cdots + a\hat{\mu}_r) \mapsto (-1)^{g_{\mu_1} + \cdots + g_{\mu_r}} \tilde{\psi}(x) M_g^\dagger \Gamma_{\mu_1 \cdots \mu_r}^m(\xi) M_g \psi(x + a\hat{\mu}_1 + \cdots + a\hat{\mu}_r) \]
\[ = \tilde{\psi}(x) \Gamma_{\mu_1 \cdots \mu_r}^m(\xi) \psi(x + a\hat{\mu}_1 + \cdots + a\hat{\mu}_r) \prod_{\mu, g, n = 1} \xi_\mu \] (26)
which is just the definition for irrep \( \rho_D^1(\xi) \) given in (15).

For diquark operators of the form
\[ \psi^T(x) \Gamma^b \psi(x + a\hat{\mu}_1 + \cdots + a\hat{\mu}_r), \quad 0 \leq r \leq 4 \]
we make use of the identity modified from (24)
\[ M_g^\dagger \gamma_{u_1} \cdots \gamma_{u_k} M_g = (-1)^{g_{u_1} + g_{u_k}} (-1)^{g_{u_1} + \cdots + g_{u_k}} \gamma_{u_1} \cdots \gamma_{u_k}, \quad \mu_1, \ldots, \mu_k \in \{1, 2, 3, 4\} \] (28)
and obtain a similar one-to-one correspondence, for each choice of links, between the gamma matrices \( \Gamma_{\mu_1 \cdots \mu_r}^b \) and the irreps \( \rho_D^1(\xi) \) of the diquark representation
\[ \Gamma_{\mu_1 \cdots \mu_r}^b(\xi) = \gamma\gamma_4 \gamma_{u_1} \cdots \gamma_{u_r} \prod_{\mu, \xi, = -1} \gamma_u \ \Leftrightarrow \ \rho_D^1(\xi) \] (29)
or explicitly
\[ \pm d(g) : \psi^T(x) \Gamma_{\mu_1 \cdots \mu_r}^b(\xi) \psi(x + a\hat{\mu}_1 + \cdots + a\hat{\mu}_r) \mapsto (-1)^{g_{\mu_1} + \cdots + g_{\mu_r}} \psi^T(x) M_g^\dagger \Gamma_{\mu_1 \cdots \mu_r}^b(\xi) M_g \psi(x + a\hat{\mu}_1 + \cdots + a\hat{\mu}_r) \]
\[ = \psi^T(x) \Gamma_{\mu_1 \cdots \mu_r}^b(\xi) \psi(x + a\hat{\mu}_1 + \cdots + a\hat{\mu}_r) \prod_{\mu, g, n = 1} \xi_\mu \] (30)

Another symmetry of the free fermion action (8) is the translational symmetry generated by single lattice spacing shifts \( t(\hat{\nu}) \) in the spatial directions given by \( \nu \in \{1, 2, 3\} \)
\[ t(\hat{\nu})(\psi(x)) = \psi(x + a\hat{\nu}) \]
\[ t(\hat{\nu})(\bar{\psi}(x)) = \bar{\psi}(x + a\hat{\nu}) \] (31)
and to be consistent with the periodic boundary conditions we must have \( t(\hat{\nu})^N \) go to identity for \( \nu \in \{1, 2, 3\} \). Therefore these shifts form a translational symmetry group \( T \)
\[ T = \{ t(n) : t(n) = \prod_{\nu} t(\hat{\nu})^{n_\nu}, n_\nu \in \mathbb{Z}_N \} \] (32)
which is an abelian group with its group operation defined as
\[ t(n^1) t(n^2) = t(n^1) t(n^2) = t(n^1 + n^2), \quad (n^1 + n^2)_\nu = n^1_\nu + n^2_\nu \ (mod N) \] (33)
and isomorphic to the direct product of cyclic group \( \mathbb{Z}_N \)
\[ T \cong (\mathbb{Z}_N)^3 \] (34)
\( T \) has \( N^3 \) 1-dimensional irreps \( \rho_D^1(p) \), labelled by a 3-component vector \( p \), the momentum
\[ p = (p_1, p_2, p_3), \quad p_\nu \in \frac{2\pi}{N}(0, \cdots, N - 1) \] (35)
such that the corresponding vector space \( \langle w(p) \rangle \) transforms under \( T \) according to
\[
\rho_T^1(p)(t(\hat{\nu})) : w(p) \mapsto e^{i\nu r}w(p) \tag{36}
\]
Define a larger lattice symmetry group \( S \) which incorporates both the doubling symmetry and the translational symmetry
\[
S = \langle d(g^\mu), t(\hat{\nu}) \rangle_{\nu \in \{1,2,3\}} \tag{37}
\]
we note its group elements \( \{ \pm d(g) \}_{g \in G} \) either commute or anticommute with \( \{ t(n) \}_{n \in \mathbb{Z}_N} \), according to
\[
t(n)d(g)t(n)^{-1} = (-1)\sum_{n \in \mathbb{Z}_N} g d(g)
\]
and therefore deduce the following properties of \( S \)
1. \( Z(S) = \{ \pm t(n) \}_{n \in \mathbb{Z}_N} \cup \{ \pm d(g^4)t(n) \}_{n \in \mathbb{Z}_N} \)
2. \( S/\{ \pm IS \} \cong G \times T \)
3. \( S = D \times T \)
By property (1) we know \( S \) breaks into \( (16 + \frac{1}{4})N^3 \) conjugacy classes
\[
\{ \pm d(g)t(n) \}_{d(g)t(n) \in Z(S)} \cup \{ -d(g)t(n) \}_{d(g)t(n) \in Z(S)} \cup \{ d(g)t(n) \}_{d(g)t(n) \in Z(S)} \tag{39}
\]
By property (2) we get \( 16N^3 \) 1-dimensional irreps of \( S \) lifted up from the 1-dimensional irreps of the abelian group \( G \times T \) in a similar fashion as in our treatment of group \( D \). Explicitly, the \( 16N^3 \) irreps of \( G \times T \) are just the tensor products of irreps of \( G \) and irreps of \( T \), labelled by a pair of vectors \( (\xi, \nu) \)
\[
\rho_T^1 \otimes \rho_G^1 = \rho_{G \times T}^1(\xi, \nu) : w(\xi, \nu) \mapsto w(\xi, \nu) \prod_{\nu \cdot g = \nu} \xi \mu \gamma \]
and the corresponding 1-dimensional irreps of \( S \), \( \rho_S^1(\xi, \nu) \), have characters
\[
\begin{array}{c|ccc}
& \{ \pm d(g)t(n) \}_{d(g)t(n) \notin Z(S)} & \{ -d(g)t(n) \}_{d(g)t(n) \in Z(S)} & \{ d(g)t(n) \}_{d(g)t(n) \in Z(S)} \\
\uparrow & \downarrow & \downarrow & \downarrow \\
\rho_S^1(\xi, \nu) & \prod_{\nu \cdot g = \nu} \xi \mu \gamma \]
\end{array}
\]
To find the remaining \( \frac{1}{4}N^3 \) irreps of \( S \), we make use of relation (38), which gives a homomorphism \( \theta \) from \( T \) to the group of automorphisms of \( D \)
\[
\theta(t(n)) : \pm d(g) \mapsto \pm t(n)d(g)t(n)^{-1} \tag{41}
\]
Since \( D \) has a single 4-dimensional irrep, \( \rho_D^4 \), given \( t(n) \) we have \( \rho_D^4 \circ \theta(t(n)) \) as a 4-dimensional irrep of \( D \) equivalent to \( \rho_D^4 \), and some \( 4 \times 4 \) transformation matrix \( P(t(n)) \) such that
\[
P(t(n))\rho_D^4(\pm d(g))P(t(n))^{-1} = \rho_D^4 \circ \theta(t(n))(\pm d(g)) \iff P(t(n))M_nP(t(n))^{-1} = (-1)^{n}M_n \tag{42}
\]
whereby \( n_4 \) is taken to be identically 0. By Schur’s lemma, the transformation matrices \( P(t(n)) \) form a projective representation of \( T \), and in particular the choice \( \phi_T^\nu \) defined as
\[
\phi_T^\nu(t(\hat{\nu})) = \gamma \nu \\
\phi_T^\nu(t(n)) = \prod_{\nu \cdot g = \nu} \gamma \nu
\]
has its factor set \( f_T \) taking values in \( \{ \pm 1 \} \)
\[
\phi_T^\nu(t(n))\phi_T^\nu(t(n')) = f_T(t(n'), t(n'))\phi_T^\nu(t(n)t(n')) \tag{43}
\]
By property (3) \( S \) has the structure
\[
S = \{ \pm d(g)t(n) : d(g^t(n'))d(g^t(n'))d(g^t(n')) = d(g^t(n))d(g^t(n))d(g^t(n)) \}
\]
therefore by (42) the map \( \phi_S^\nu \) defined as
\[
\phi_S^\nu(\pm d(g)t(n)) = \rho_D^4(\pm d(g))\phi_T^\nu(t(n)) \tag{46}
\]
is a projective representation of \( S \), with its factor set \( f_S \) inherited from that of \( g_1^2 \)

\[
f_S(\pm d(g^i)t(n^i), \pm d(g^j)t(n^j)) = f_T(t(n^i), t(n^j)) \tag{47}
\]

Since the 4-dimensional projective representation of \( T \), \( g_2^T \), given by (43) maps the generators of the abelian group, \( \{t(\nu)\}_{\nu \in \{1, 2, 3\}} \), to anticommuting matrices, we are inspired to find a 2-dimensional projective representation of \( T \), \( g_2^T \), defined as

\[
\begin{align*}
\rho_2^T(t(\nu)) &= \sigma_\nu, \\
\rho_2^T(t(n)) &= \prod_\nu \sigma_\nu^{n_\nu} = \prod_{\nu : n_\nu \notin 2Z_N} \sigma_\nu
\end{align*}
\tag{48}
\]

such that it has the same factor set as that of \( g_2^1 \), \( f_T \). By property (3) we also have

\[
q : S \to S/D \cong T \tag{49}
\]

therefore we can lift \( g_2^2 \) up to the corresponding 2-dimensional projective representation of \( S \), \( \tilde{g}_S^2 \), with the same factor set as that of \( g_2^1 \), \( f_S \). Similarly we can lift \( \rho_2^T(p) \) up to \( \rho_S^2(p) \) as a 1-dimensional true representation of \( S \). Finally, we define an 8-dimensional projective representation of \( S \), \( \tilde{g}_S^8 \), as the tensor product

\[
\rho_S^8(p) = \rho_S^2(p) \otimes g_2^1 \otimes g_1^3 \tag{50}
\]

and see that it has the trivial factor set \( f_S^8 \)

\[
f_S^8(\pm d(g^i)t(n^i), \pm d(g^j)t(n^j)) = f_S(\pm d(g^i)t(n^i), \pm d(g^j)t(n^j))^2 \equiv 1 \tag{51}
\]

and is thus in fact a true representation of \( S \). Explicitly, we have the group elements of \( S \) represented as the \( 8 \times 8 \) matrices

\[
\rho_S^8(p)(\pm d(g)t(n)) = \pm e^{i \sum_{\nu : n_\nu \notin 2Z_N} \sigma_\nu \otimes M_{g} \prod_{\nu : n_\nu \notin 2Z_N} \gamma_\nu} \tag{52}
\]

and taking the product of traces of the matrices \( \rho_S^8(\pm d(g)t(n)) \) and \( \rho_S^2(\pm d(g)t(n)) \), we obtain the character of \( \rho_S^8(p) \) as

\[
\begin{array}{c|ccc}
\rho_S^8(p) & 0 & -8e^{i \sum_{\nu : n_\nu \notin 2Z_N} \sigma_\nu} & 8e^{i \sum_{\nu : n_\nu \notin 2Z_N} \sigma_\nu} \\
\hline
\{\pm d(g)t(n)\}_{d(g)t(n) \notin Z(S)} & \{d(g)t(n)\}_{d(g)t(n) \in Z(S)} & \{d(g)t(n)\}_{d(g)t(n) \in Z(S)}
\end{array}
\]

\( \rho_S^8(p) \) is irreducible as its character obeys the relation

\[
\langle \chi_{\rho_S^8(p)}, \chi_{\rho_S^8(p)} \rangle = 1 \tag{53}
\]

For each \( p \) in the range given by \( p_\nu \in \frac{2\pi}{N} \{0, \cdots, \frac{N}{2} - 1\} \), define a new 8-dimensional irrep of \( S \) from \( \tilde{\rho}_S^8(p) \), denoted by \( \bar{\rho}_S^8(p) \), as

\[
\tilde{\rho}_S^8(p) = \rho_S^8(\tilde{\pi}) \otimes \rho_S^8(p) \tag{54}
\]

whereby \( \tilde{\pi} \) denotes the 3-component vector \((\pi, \pi, \pi)\). These \( \frac{1}{2}N^3 \) irreps

\[
\{\rho_S^8(p)\} \cup \{\bar{\rho}_S^8(p)\}, \quad p_\nu \in \frac{2\pi}{N} \{0, \cdots, \frac{N}{2} - 1\} \tag{55}
\]

give distinct characters and are thus pairwise inequivalent. We therefore exhaust the remaining irreps of \( S \), and obtain the full character table of group \( S \) as follows

\[
\begin{array}{c|ccc}
\hline
\rho_S^8(\kappa, p) & \{\pm d(g)t(n)\}_{d(g)t(n) \notin Z(S)} & \{d(g)t(n)\}_{d(g)t(n) \in Z(S)} & \{d(g)t(n)\}_{d(g)t(n) \in Z(S)} \\
\hline
\prod_{\nu : g_\nu = 1} \xi_\nu e^{i \sum_{\nu : n_\nu \notin 2Z_N} \sigma_\nu} & \prod_{\nu : g_\nu = 1} \xi_\nu e^{i \sum_{\nu : n_\nu \notin 2Z_N} \sigma_\nu} & \prod_{\nu : g_\nu = 1} \xi_\nu e^{i \sum_{\nu : n_\nu \notin 2Z_N} \sigma_\nu} \\
\hline
\rho_S^8(\tilde{\pi}) & 0 & -8e^{i \sum_{\nu : n_\nu \notin 2Z_N} \sigma_\nu} & 8e^{i \sum_{\nu : n_\nu \notin 2Z_N} \sigma_\nu} \\
\hline
\rho_S^8(\tilde{\pi}) & 0 & -8(-1) e^{i \sum_{\nu : n_\nu \notin 2Z_N} \sigma_\nu} & 8(-1) e^{i \sum_{\nu : n_\nu \notin 2Z_N} \sigma_\nu}
\end{array}
\]
with \( p \) in the range given by \( p_\nu \in \frac{2\pi}{N} \{0, \ldots, N - 1\} \) and \( \hat{p} \) in the range given by \( \hat{p}_\nu \in \frac{2\pi}{N} \{0, \ldots, N - 1\} \). For each \( \hat{p} \), both \( \rho_S^\delta (\hat{p}) \) and \( \tilde{\rho}_S^\delta (\hat{p}) \) give an 8-dimensional representation of \( T \) as a subgroup of \( S \), denoted by \( \rho_S^\delta (\hat{p}) | T \) and \( \tilde{\rho}_S^\delta (\hat{p}) | T \) respectively. Modifying (52) we have

\[
\rho_S^\delta (\hat{p}) | T (t(n)) = e^{i \sum \nu n_\nu \hat{p}_\nu} \prod_{\nu: n_\nu \not\in 2Z_N} \sigma_\nu \otimes \prod_{\nu: n_\nu \not\in 2Z_N} \gamma_\nu \\
\tilde{\rho}_S^\delta (\hat{p}) | T (t(n)) = (-1)^{\sum \nu n_\nu} e^{i \sum \nu \hat{p}_\nu} \prod_{\nu: n_\nu \not\in 2Z_N} \sigma_\nu \otimes \prod_{\nu: n_\nu \not\in 2Z_N} \gamma_\nu
\]

(56)

and employing the projection formulas for the multiplicities of irreps of \( T \) in \( \rho_S^\delta (\hat{p}) | T \) and \( \tilde{\rho}_S^\delta (\hat{p}) | T \)

\[
m_{\rho_S^\delta (\hat{p}) | T} = \langle \rho_S^\delta (\hat{p}), \chi_{\rho_S^\delta (\hat{p})} \rangle = \frac{8}{N^3} \prod_{\nu} \sum_{n_\nu \in 2Z_N} e^{in_\nu (\hat{p}_\nu - p_\nu)} \\
m_{\tilde{\rho}_S^\delta (\hat{p}) | T} = \langle \tilde{\rho}_S^\delta (\hat{p}), \chi_{\tilde{\rho}_S^\delta (\hat{p})} \rangle = \frac{8}{N^3} \prod_{\nu} \sum_{n_\nu \in 2Z_N} e^{in_\nu (\hat{p}_\nu - p_\nu)}
\]

(57)

we deduce

\[\rho_S^\delta (\hat{p}) | T \cong \tilde{\rho}_S^\delta (\hat{p}) | T = \oplus_{p,p+\hat{p} = 0,\pi} \rho_S^\delta (p) \]

(58)

whereby \( \pi \) denotes a 3-component vector such that \( \pi_\nu \in \{0, \pi\} \). In other words, given \( \hat{p} \) with \( \hat{p}_\nu \in \frac{2\pi}{N} \{0, \ldots, N - 1\} \), the two 8-dimensional irreps of \( S \) associated with \( \hat{p}, \rho_S^\delta (\hat{p}) \) and \( \tilde{\rho}_S^\delta (\hat{p}) \), both couple to the eight 1-dimensional irreps of \( T \), \( \{\rho^\delta_k (\hat{p} + \pi)\}_{\nu \in \{0,\pi\}} \), but are nonetheless inequivalent representations of the whole group \( S \) taking into account the doubling symmetry.

On the quark field, the identification of \(-I_S \to I_S \) in the 1-dimensional irreps of \( S \), \( \rho^\delta_k (\xi, p) \), is unnatural, whereby we conclude that the representation of \( S \) on the quark field decomposes into copies of \( \rho_S^\delta (\hat{p}) \) and \( \tilde{\rho}_S^\delta (\hat{p}) \) for different \( \hat{p} \) in the range given by \( \hat{p}_\nu \in \frac{2\pi}{N} \{0, \ldots, N - 1\} \). By the same reasoning the representation of \( S \) on the antiquark field decomposes into copies of the 8-dimensional irreps of \( S \) alike.

To analyse the diquark representation and the antiquark-quark representation, it suffices to look at the four cases, \( \rho_S^\delta (\hat{p}) \otimes \rho_S^\delta (\hat{p}') \), \( \rho_S^\delta (\hat{p}) \otimes \tilde{\rho}_S^\delta (\hat{p}') \), \( \tilde{\rho}_S^\delta (\hat{p}) \otimes \rho_S^\delta (\hat{p}') \), and \( \tilde{\rho}_S^\delta (\hat{p}) \otimes \tilde{\rho}_S^\delta (\hat{p}') \), for a particular choice of \( \hat{p}' \) and \( \hat{p} \). Taking the products of characters of the factors we compute the characters of these four representations as follows

\[
\begin{array}{c|c|c|c}
\{\pm d(g)(t(n))\}_{d(g)t(n) \not\in Z(S)} & \{-d(g)(t(n))\}_{d(g)t(n) \in Z(S)} & \{d(g)(t(n))\}_{d(g)t(n) \in Z(S)} \\
\rho_S^\delta (\hat{p}) \otimes \rho_S^\delta (\hat{p}') & 0 & 64 e^{i \sum \nu n_\nu (\hat{p}'_\nu + \hat{p}_\nu)} & 64 e^{i \sum \nu n_\nu (\hat{p}'_\nu + \hat{p}_\nu)} \\
\rho_S^\delta (\hat{p}) \otimes \tilde{\rho}_S^\delta (\hat{p}') & 0 & 64 (-1)^{\sum \nu n_\nu} e^{i \sum \nu n_\nu (\hat{p}'_\nu + \hat{p}_\nu)} & 64 (-1)^{\sum \nu n_\nu} e^{i \sum \nu n_\nu (\hat{p}'_\nu + \hat{p}_\nu)} \\
\tilde{\rho}_S^\delta (\hat{p}) \otimes \rho_S^\delta (\hat{p}') & 0 & 64 (-1)^{\sum \nu n_\nu} e^{i \sum \nu n_\nu (\hat{p}'_\nu + \hat{p}_\nu)} & 64 (-1)^{\sum \nu n_\nu} e^{i \sum \nu n_\nu (\hat{p}'_\nu + \hat{p}_\nu)} \\
\tilde{\rho}_S^\delta (\hat{p}) \otimes \tilde{\rho}_S^\delta (\hat{p}') & 0 & 64 (-1)^{\sum \nu n_\nu} e^{i \sum \nu n_\nu (\hat{p}'_\nu + \hat{p}_\nu)} & 64 (-1)^{\sum \nu n_\nu} e^{i \sum \nu n_\nu (\hat{p}'_\nu + \hat{p}_\nu)} \\
\end{array}
\]

Denoting

\[
\rho_S^\delta (\hat{p}') \otimes \rho_S^\delta (\hat{p}) \equiv \tilde{\rho}_S^\delta (\hat{p}') \otimes \tilde{\rho}_S^\delta (\hat{p}) = \zeta (\hat{p}', \hat{p}) \\
\rho_S^\delta (\hat{p}') \otimes \tilde{\rho}_S^\delta (\hat{p}) \equiv \rho_S^\delta (\hat{p}') \otimes \rho_S^\delta (\hat{p}) = \zeta'(\hat{p}', \hat{p})
\]

(59)

we have the overlaps between \( \zeta(\hat{p}', \hat{p}) \), \( \zeta'(\hat{p}', \hat{p}) \) and the 1-dimensional irreps of \( S \), \( \rho_S^\delta_k (\xi, p) \), to be

\[
m_{\rho_S^\delta (\hat{p}') \otimes \rho_S^\delta (\hat{p})} = \langle \rho_S^\delta (\hat{p}', \xi), \chi_{\rho_S^\delta (\hat{p}, \xi)} \rangle = \frac{4}{N^3} (1 + e^{i \sum \nu n_\nu (\hat{p}'_\nu + \hat{p}_\nu - p_\nu)}) \prod_{\nu} \sum_{n_\nu \in Z_N} e^{in_\nu (\hat{p}'_\nu + \hat{p}_\nu - p_\nu)} \\
m_{\rho_S^\delta (\hat{p}') \otimes \tilde{\rho}_S^\delta (\hat{p})} = \langle \rho_S^\delta (\hat{p}', \xi), \chi_{\tilde{\rho}_S^\delta (\hat{p}, \xi)} \rangle = \frac{4}{N^3} (1 + e^{i \sum \nu n_\nu (\hat{p}'_\nu + \hat{p}_\nu - p_\nu)}) \prod_{\nu} \sum_{n_\nu \in Z_N} e^{in_\nu (\hat{p}'_\nu + \hat{p}_\nu - p_\nu)}
\]

(60)

therefore \( \zeta(\hat{p}', \hat{p}) \) and \( \zeta'(\hat{p}', \hat{p}) \) each reduce to 64 1-dimensional irreps of \( S \) associated with \( \hat{p}' + \hat{p} \) according to

\[
\zeta(\hat{p}', \hat{p}) = \oplus_{p,p=\hat{p}'+\hat{p}+\pi} \rho_S^\delta_k (\xi, p) \\
\zeta'(\hat{p}', \hat{p}) = \oplus_{p,p=\hat{p}'+\hat{p}+\pi} \rho_S^\delta_k (\xi, p)
\]

(61)

while the two sets of irreps from \( \{\rho_S^\delta_k (\xi, p)\}_{p=\hat{p}'+\hat{p}+\pi} \) do not overlap with each other.