Natural Convection in Melting Icicles

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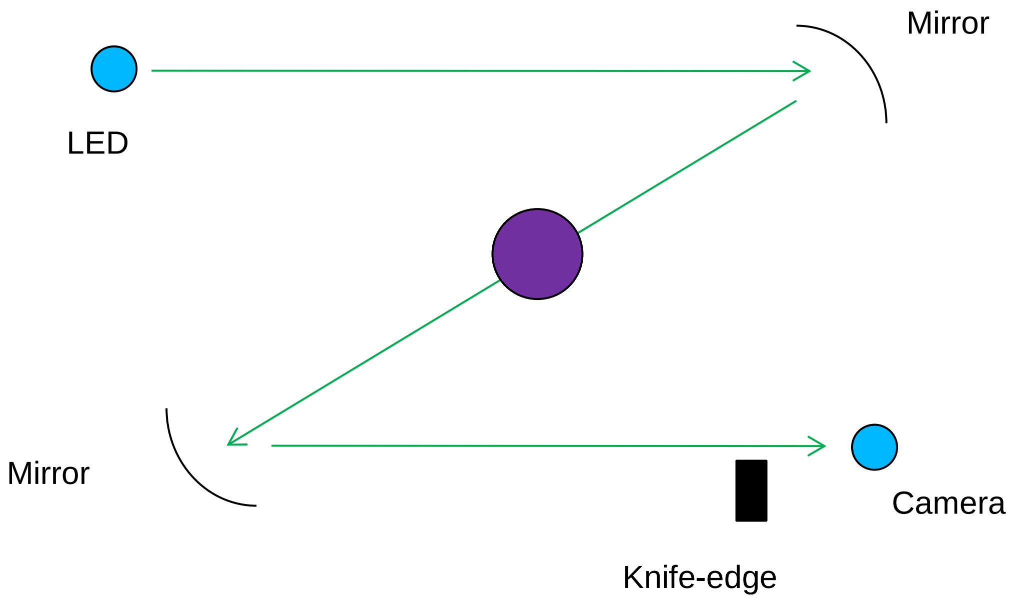
Introduction

The aim of this project was to understand the physical principles involved in the melting of an icicle and to develop a model that is able to predict the shape evolution of an icicle with arbitrary initial shape. The project comprised both an experimental and a theoretical part. The experimental aspect consisted in using Schlieren photography to resolve the thin convective boundary layers that exist in the vicinity of the surface of the ice. The theoretical piece focused on using elements of boundary layer theory and convective heat transfer to find governing equations for the problem and attempting to solve these numerically.

This work was undertaken by myself and Anthony Fragoso, an undergraduate at Yale University, with the support of our project supervisors Jerome Neufeld (DAMTP) and Grae Worster (DAMTP).

Experiment

As already mentioned in the introduction, we made use of Schlieren photography to visualise the convective boundary layers that occur on the surface of the icicle. The setup of the experiment is shown below (in plan view).



Plan view of
Experimental Set-up

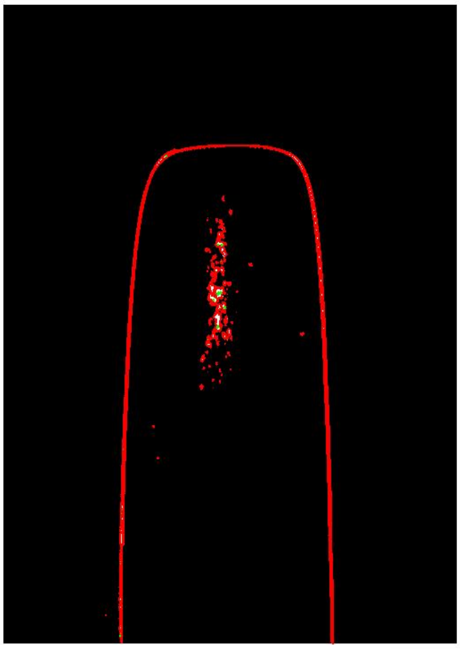
The aim of the experimental method is to obtain data corresponding to the variation in ice shape and boundary layer height during a complete melting cycle of a right circular cylinder of ice.

Description of Schematic: Light from an LED hits the focal point of a parabolic mirror, the light is then reflected past the ice cylinder (in this region refraction occurs because of the varying temperature gradients which cause varying density gradients). The beam is then reflected at another parabolic mirror and the image is recorded by a digital camera.

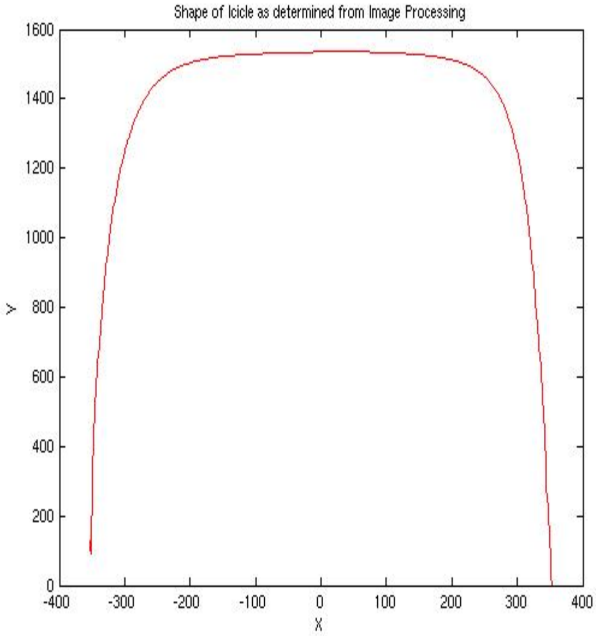
Schlieren is able to resolve the temperature gradients, it does so by obstructing some of the light that comes from the object being observed: we use a knife-edge (physically, a razor blade) to 'cut-off' high spatial frequency components. The only problem is that using a knife-edge means that we are only able to resolve the boundary layers in one direction; hence in one photograph we only get vertical or horizontal boundary layers. What I was therefore trying to implement in the last week of the project was a "Pinhead Schlieren" method, where the knife-edge is replaced by a pinhead. This meant being able to resolve the boundary layers in any direction. However, after running an experiment with this method we realised that the images were of much lower quality than the ones obtained using the knife-edge. In my final days at DAMTP I tried a different method- using a wedge- in order to obtain the vertical and horizontal boundary layers, but again after running an experiment we realised that this was unsuccessful in providing useful data. In both the pinhead and the wedge I think that the reason for the bad results was that too much light was being obstructed, i.e. the radius of the pinhead was not well-suited to the size of the beam of light.

After having obtained images, we made use of image processing techniques to plot the ice shape as a smooth curve, with the boundary layer superimposed on it. All of this part of the project was done using Matlab code. Sample results are shown below for a melting icicle after approximately three hours.

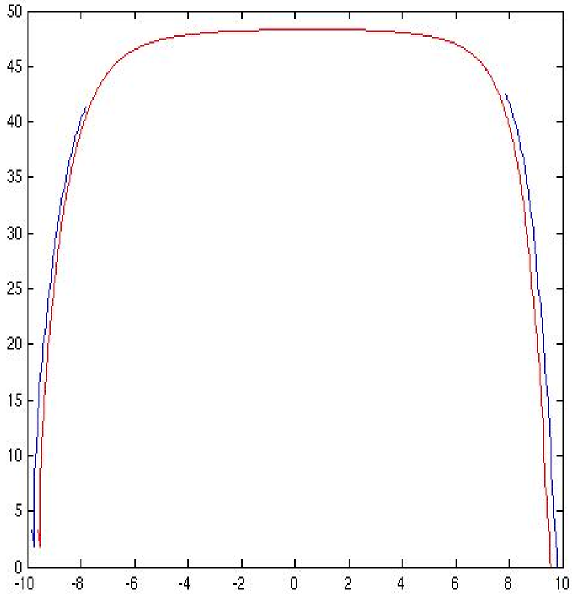
Raw Image



Boundary tracing after conversion to black & white



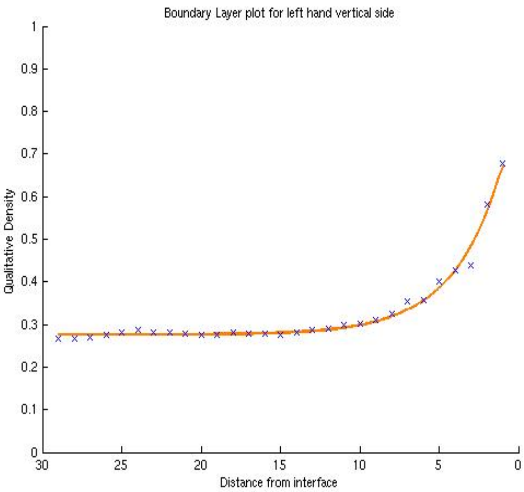
Retrieved Ice Shape



With Boundary Layer Superimposed

The **method** used to obtain the boundary layer is as follows:

1. We range along the ice edge (going from one coordinate pair to the next- note that although the curve is smooth the data is still discrete). Fix that point.



2. If that point is on one of the vertical sides then range horizontally (range vertically if the point is on the horizontal side). Plot the pixel intensity variation in the greyscale conversion of the image as a function of distance (along normal) from the interface. Then, we note that the decay in intensity normally away from the surface of the icicle is exponential. We therefore fit a curve and make the experimentally motivated assumption for the functional form of the temperature decay (written below to the left).

$$T = T_\infty\left(1 - e^{-\frac{z}{h(x)}}\right)$$

3. Finally, by matching the curve fitting coefficients to the assumed profile we are able to extract the value of the boundary layer height as a function of arc length away from the tip of the icicle; a variable which we call x.

Theory

The governing equations for our problem are shown below.

$$v\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_\infty}\frac{\partial P}{\partial x} + g\alpha(T - T_\infty)\sin\varphi + v\frac{\partial^2 u}{\partial z^2}$$

$$0 = -\frac{1}{\rho_\infty}\frac{\partial P}{\partial z} + g\alpha(T - T_\infty)\cos\varphi$$

$$u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = \kappa\frac{\partial^2 T}{\partial z^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\rho = \rho_\infty[1 - \alpha(T - T_\infty)]$$

The first two equations are Navier Stokes in two dimensions (this is enough to describe the problem in three dimensions since we assume axisymmetry). The third equation is the heat equation, representing the usual conservation of heat. The fourth is incompressibility and the final equation is the usual gas law used to model an ideal gas such as air.

Our task is to solve the above system of PDE's. To simplify the job we need to assume suitable profiles for the above functions, motivated by the physics governing the situation. These are

$$T(x,z) = T_\infty + \Delta T\theta(\eta)$$

$$\psi(x,z) = \kappa q(x)f(\eta)$$

$$P(x,z) = \rho_\infty\alpha\Delta T g h(x)p(\eta)$$

Where we have introduced the similarity variable $\eta = \frac{z}{h(x)}$

These are the same profiles introduced in the paper published by Goldstein, Neufeld & Worster (2000).

Where the z coordinate is displacement normally away from the surface of the ice, and the angle φ is the angle between the horizontal and the tangent vector.

Then, we can use an integral representation of the boundary layer equations to proceed, i.e. we integrate the whole equation in the z direction from 0 to h(x). In order to do this we have to assume a temperature variation (we use the same experimentally motivated profile as shown in the previous section). And using thin-layer (lubrication theory) approximation that the flow is parabolic in the boundary layer (this is the usual gravity current analysis with zero shear stress on the free surface)- see below.

$$u = u_\infty \left(1 - \frac{z}{h(x)}\right)^2$$

Having done this it is possible to carry out the integration and to obtain a differential equation in h(x).

After suitable non-dimensionalisation the equation reduces to the gravity current equation for an inclined plate. And this makes physical sense.

(1) $$\boxed{(-h^3 h' \cos\varphi + h^3 \sin\varphi)' = -\frac{1}{h}}$$

Solving this then allows us to solve the system of ODE’s that we obtain from plugging the assumed profiles into the PDE’s. Having done this, we have an expression for the temperature gradient in the normal direction, and thus using the Stefan condition, an expression for the normal melt rate (written v subscript n).

Using a Langer argument analogous to his ‘String Model’ in his 1987 paper we can find the local curvature (H) and thus φ. This allows us to predict the shape evolution of the melting icicle.

(2) $$\boxed{\frac{dH}{dt} = -\left(\frac{\partial^2}{\partial x^2} + H^2\right) v_{\mathrm{n}} - \frac{\partial H}{\partial x}\int_0^x H v_{\mathrm{n}}\, dx'}$$

We have yet to solve equation (1) numerically, and this summarises at which stage we are in the theoretical solution of the problem.

Conclusions

Overall, the aim was to compare the theoretical model with experiment, which we are not in a position to do at this present time. However, even though we are not able to fully assess the validity of the theory by direct comparison with experiment I have thoroughly enjoyed working on this project and have learned much interesting physics. I would like to thank my supervisors and sponsors for enabling me to participate in this project, hosted by DAMTP.

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