**A Multiplicative Regularization Approach for Inverse Problems in Image Processing**

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**Problem Statement**

Noisy, blurred image can be described as $f = \mathbb{K}u + \varepsilon$, where $u = u(x)$ denotes the original image to recover, $\varepsilon$ is the random noise (unknown), $f$ is the corrupted image (observation), and $\mathbb{K}$ is the blurring operator. The task is to reconstruct $u$ from known $\mathbb{K}$ and $f$. This problem can be turned into a minimization problem, and the cost functional to minimize is $C(u) = F(u) = \| \mathbb{K}u - f \|_D$

Blurring operator $\mathbb{K}$ is an integral operator of form $[\mathbb{K}u](x) = \int_D K(x - x')u(x')dx$

$\| \cdot \|_D$ denotes the $L_2$ norm on domain $D$, $K(x - x')$ is the blurring kernel.

However, since blurring operator $\mathbb{K}$ is compact (bounded in $R^2$), the problem of reconstructing $u$ by minimizing $C(u) = F(u) = \| \mathbb{K}u - f \|_D$ is ill-posed (it inverts a compact operator directly). The example below demonstrates that this naive idea will not work.

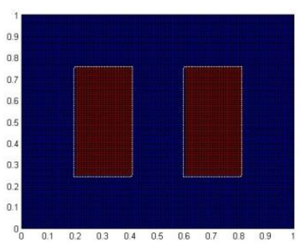

Fig 1. Original image to reconstruct

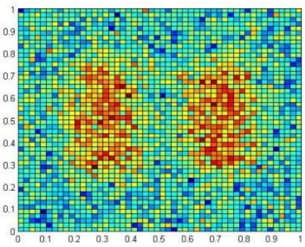

Fig 2. Image corrupted by 30% Gaussian Noise

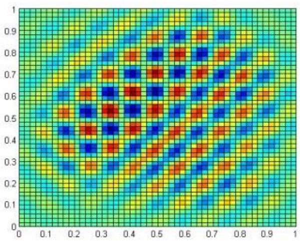

Fig 3. Restored Image

To solve this problem, we need to introduce a penalty term to the cost functional we minimize to restore the stability of the method. This technique is called Regularization, which tackles inverse problems specifically.

**Classical Approach: Additive Type of Regularization**

The classical regularization approach *adds* a small penalty term to the functional $F(u) = \| \mathbb{K}u - f \|_D$ we minimized before. The cost functional to minimize is modified into

$C(u) = F(u) + \alpha F^R(u)$, where $F(u) = \| \mathbb{K}u - f \|_D$, and $F^R(u) = \int_D (| \nabla u(x) |^2 + \delta^2)^{\frac{1}{2}} dx$

Here $\alpha$ is called regularization parameter. It is a small and positive real number that controls the regularization strength. The choice for $\alpha$ is crucial: if it is too small, the regularization effect will be too weak to stabilize the problem; while if it is too large, the problem might change nature and causes the algorithm to converge to the wrong image.

Use the same example as above, the following images are constructed with different $\alpha$

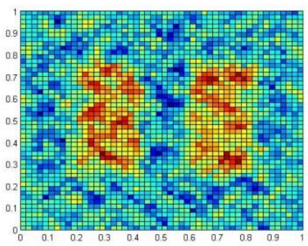

Fig 4. Restored image with small $\alpha$

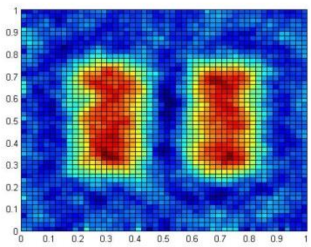

Fig 5. Restored image with proper $\alpha$

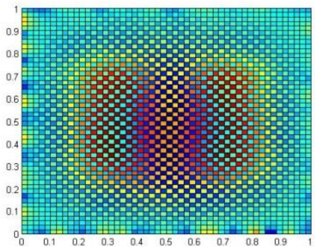

Fig 6. Restored image with large $\alpha$

In Practice it is usually very time-consuming and computationally costly to determine the sensible value for $\alpha$ . Extra analysis is required. This is the motivation for a new type of regularization: The *Multiplicative* type of regularization.

## New Approach: Multiplicative Type of Regularization

In this new approach, instead of *adding* a penalty term, we multiply the original functional $F(u)$ with regularization factor $F^R(u)$ . The cost functional to minimize is $C(u) = F(u)F^R(u)$, where $F(u) = \| \mathbb{K}u - f \|_D$, $F^R(u) = \int_D b(x)^2 (|\nabla u(x)|^2 + \delta^2)\ dx$

Here weight function $b(x)$ plays the role of regularization parameter $\alpha$ in the additive cost functional: it controls the strength of regularization. The parameter $\delta$ has two functions: to ensure the positiveness of $F^R(u)$ and to ensure the convexity of the highly non-linear cost functional $C(u)$ . In the algorithm, $b(x)$ and $\delta$ are defined iteratively in each loop: $b_n = V^{-\frac{1}{2}} (|\nabla u_{n-1}(x)|^2 + \delta_{n-1}^2)^{-\frac{1}{2}}$ and $\delta_n^2 = \frac{1}{2} \frac{\| b_n \nabla u_{n-1} \|_D^2}{\| b_n \|_D^2}$ .This new approach of regularization helps to get rid of the trouble of deciding regularization parameter totally. No extra analysis is required. The restored images using Additive and Multiplicative regularization are presented below.

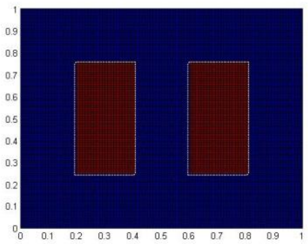

Fig 7. Original Image

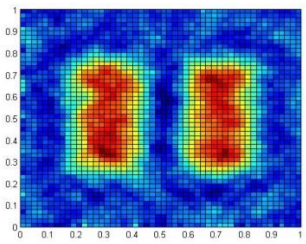

Fig 8. Restored image (Additive type )

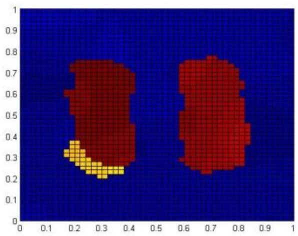

Fig 9. Restored Image (Multiplicative Type)

## Conclusion

Comparing these two approaches of regularization, we may conclude that in the case shown the multiplicative type works better than the additive type. This is because the cost functional $C(u)$ changes in the multiplicative type regularization. As the number of iterations increases (n increases), $C(u) \to F(u)$ and the regularization effect gets weaker and weaker. We can see that the reconstructed solution gets closer and closer to the exact solution. The new functional and new iterate at each step produce a calculation that becomes less nonlinear, getting closer and closer to quadratic minimization. This allows more accurate reconstruction and thence better result is produced.

## Further Work

The multiplicative type of regularization is a relatively new approach and not much research has been done on it. So far it is not very clear whether this algorithm will guarantee convergence or not. There are classical results about convergence on conjugate gradient methods and they work very well for minimization problems where the cost functional to minimize is given and fixed over all iterations. It is interesting and worth finding out whether these results can be extended to cases where the cost functional to minimize changes in each loop.

## References

A.Abubakar, P.M.van den Berg, T.M.Habashy and H. Braunisch "A Multiplicative Regularization Approach for Deblurring Problems," *IEEE Transactions on Image Processing*, Vol 13, No.11, November 2004

D.Strong and T.Chan, "Edge-preserving and scale-dependent properties of total variation regularization", *Institute of Physics Publishing*, Inverse Problems 19, pp165-187, 2003