Ordinary and Modular Representations of Finite Symmetric Groups

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Introduction

The symmetric groups, S_n, are probably the most important class of finite groups due to Cayely's theorem (Every group G is isomorphic to a subgroup of a symmetric group). The ordinary representation theory of S_n is relatively well understood, but the modular theory, started by Dickson in 1902 and advanced massively by Brauer in 1935, is still under development. Indeed elementary quantities such as the dimension of the irreducible p-modular representations are still widely unknown.

On this poster, we shall construct the irreducible ordinary and modular representations for the finite symmetric group. In ordinary representation theory, we can explicitly describe how the obvious permutation modules decompose into the irreducible ones and we can begin to consider how these are linked to the p-modular representation. This brings us the very rich research area of decomposition matrices. These show the factors of the irreducible ordinary representations on reduction modulo p.

Definitions

The collection of all permutations of $\{1,\ 2,\ \dots,\ n\}$ forms a group under the composition of functions. We call this group S_n and is the focus of this theory.

We say $\rho : G \to GL(V)$ is a *representation* for G if ρ is a group homomorphism (i.e $\rho(\iota) = Id_V : v \to v$ where ι is the identity in G and $\forall \sigma, \tau \in S_n, \rho(\sigma\tau) = \rho(\sigma)\rho(\tau)$). Here V is a vector space over a field F and $GL(V)$ is the group of automorphisms of V. V is often called a G-module. If the characteristic of F does not divide the order of the group, we call this an *ordinary* representation otherwise, it is a *modular* representation.

As with prime numbers within the natural numbers, it is beneficial to consider the irreducible components of these representations. We say ρ (or V) is *irreducible* if there does not exist U, a proper subspace of V, such that $\forall \sigma \in S_n\ \sigma U \subset U$.

Combinatorial Tools

An elementary result of ordinary representation theory states that the number of irreducible inequivalent representations of a group G is equal to the number of conjugacy classes for G. In the case of S_n, the conjugacy classes are characterised by the partitions of n (e.g $(4,3,3,1)$ is a partition of 11 as $4+3+3+1 = 11$, we often abbreviate this to $(4,3^2,1)$), so we should attempt to construct an irreducible FS_n-module for each partition of n.

Fix a partition μ, the *diagram* for μ is a collection of crosses showing the shape of μ. This is best described in the form of an example, the diagram of $(3,2)$ is $\begin{array}{lll} X & X & X \\ X & X & \end{array}$ and that of $(4,3^2,1)$ is $\begin{array}{llll} X & X & X & X \\ X & X & X & \\ X & X & X & \\ X & & & \end{array}$.

A μ-*tableau* is a diagram for μ where the numbers $1, 2, \dots, n$ replace the X's. Continuing the example above, $t = \begin{array}{llll} 2 & 5 & 1 & 3 \\ 8 & 7 & 4 & \\ 6 & & & \end{array}$ is a $(4,3,1)$-tableau.

Finally, a *tabloid* is a μ-tableau where the order with each row is unimportant, this is written as

$$\{t\} = \begin{array}{llll} \hline 2 & 5 & 1 & 3 \\ \hline 8 & 7 & 4 & \\ \hline 6 & & & \\ \hline \end{array} = \begin{array}{llll} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 7 & 8 & \\ \hline 6 & & & \\ \hline \end{array}.$$

Constructing Representations

We can now define the module M^μ as the vector space over a field F with basis consisting of the μ-tabloids and S_n acts on the basis of tabloids component wise, e.g. for $\pi = (1,2)(3,4) \in S_5$,

$$\pi\begin{pmatrix} \begin{array}{lll} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array} \end{pmatrix} = \begin{array}{lll} \hline \pi(1) & \pi(2) & \pi(3) \\ \hline \pi(4) & \pi(5) & \\ \hline \end{array} = \begin{array}{lll} \hline 2 & 1 & 4 \\ \hline 3 & 5 & \\ \hline \end{array} = \begin{array}{lll} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline \end{array}.$$

Alternatively, this can be thought of as the permutation module of S_n acting on the cosets of the Young subgroup $S_\mu := S_{\mu_1} \times S_{\mu_2} \times \dots \times S_{\mu_k}$ where $\mu = (\mu_1, \mu_2, \dots, \mu_k)$. In most cases this is not irreducible, indeed the subspace generated by the sum of the μ-tabloids is a one dimensional subspace, on which S_n acts trivially.

We now seek a particular type of submodule, called the *Specht module*. Define $\kappa_t = \sum(sign\ \pi)\pi$ where the sum is taken over the elements of the column stabiliser of t. A *polytabloid* is defined as $e_t = \kappa_t\{t\} \in M^\mu$. For example, if $t = \begin{array}{lll} 1 & 2 & 3 \\ 4 & 5 & \end{array}$, then $\kappa_t = \iota - (1,4) - (2,5) + (1,4)(2,5)$ and

$$e_t = \begin{array}{lll} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array} - \begin{array}{lll} \hline 4 & 2 & 3 \\ \hline 1 & 5 & \\ \hline \end{array} - \begin{array}{lll} \hline 1 & 5 & 3 \\ \hline 4 & 2 & \\ \hline \end{array} + \begin{array}{lll} \hline 4 & 5 & 3 \\ \hline 1 & 2 & \\ \hline \end{array}.$$

Notice all the coefficients of the tabloids in e_t are ± 1 and $\forall \pi \in S_n,\ \pi e_t = e_{\pi t}$. Define S^μ to be the vector subspace of M^μ generated by these polytabloids. The previous comments show that S^μ is actually a submodule of M^μ generated by any one polytabloid.

To deal with irreducibility, we require the Submodule Theorem: If U is a submodule of M^μ, the either $U \supset S^\mu$ or $U \subset S^{\mu\perp}$, where $S^{\mu\perp}$ is the orthogonal complement with respect to the bilinear form $< t_1, t_2 >= 1$ if $\{t_1\} = \{t_2\}$ and 0 otherwise. An immediate corollary of this is that the quotient module, $\frac{S^\mu}{S^\mu \cap S^{\mu\perp}}$, is zero or irreducible.

Ordinary Representations

In the ordinary case, $< *, * >$ is an inner product so $S^\mu \cap S^{\mu\perp} = \{0\}$ and S^μ is irreducible. It can be shown that $S^\mu \cong S^\lambda$ if and only if $\mu = \lambda$ so these form a complete set of irreducible ordinary modules for S_n. As with other permutation modules, M^μ and S^λ depend only on the prime subfield (ie the subfield generated by the 1).

For the characteristic zero case it suffices to only consider the field $\mathbb{Q}$. In this case, the decomposition of M^μ into the Specht modules S^λ is given by Young's rule: the multiplicity of S^λ as a factor of M^μ is the number of semistandard λ-tableaux of type μ, where a tableau has *type* $\mu = (\mu_1,\ \mu_2,\ \dots, \mu_k)$ if the tableau contains μ_1 1's, μ_2 2's, etc. It is *semistandard* if the numbers strictly increase down the columns and are non-decreasing along the rows. This reduces the complex algebraic questions of decomposition into a simple case of counting.

As an example, we shall calculate the multiplicity of $S^{(4,\ 3^2,\ 1)}$ as a factor of $M^{(3,\ 2^4)}$. By Young's rule, we seek semistandard $(4,\ 3^2,\ 1)$-tableau of type $(3,\ 2^4)$, these are listed below:

$$\begin{array}{llll} 1 & 1 & 1 & 2 \\ 2 & 3 & 3 & \\ 4 & 4 & 5 & \\ 5 & & & \end{array},\quad \begin{array}{llll} 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & \\ 3 & 4 & 5 & \\ 5 & & & \end{array},\quad \begin{array}{llll} 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & \\ 3 & 5 & 5 & \\ 4 & & & \end{array},\quad \begin{array}{llll} 1 & 1 & 1 & 3 \\ 2 & 2 & 3 & \\ 4 & 4 & 5 & \\ 5 & & & \end{array},\quad \begin{array}{llll} 1 & 1 & 1 & 3 \\ 2 & 2 & 4 & \\ 3 & 4 & 5 & \\ 5 & & & \end{array},\quad \begin{array}{llll} 1 & 1 & 1 & 3 \\ 2 & 2 & 4 & \\ 3 & 5 & 5 & \\ 4 & & & \end{array},$$

$$\begin{array}{llll} 1 & 1 & 1 & 4 \\ 2 & 2 & 3 & \\ 3 & 4 & 5 & \\ 5 & & & \end{array},\quad \begin{array}{llll} 1 & 1 & 1 & 4 \\ 2 & 2 & 3 & \\ 3 & 5 & 5 & \\ 4 & & & \end{array},\quad \begin{array}{llll} 1 & 1 & 1 & 4 \\ 2 & 2 & 4 & \\ 3 & 3 & 5 & \\ 5 & & & \end{array},\quad \begin{array}{llll} 1 & 1 & 1 & 5 \\ 2 & 2 & 3 & \\ 3 & 4 & 4 & \\ 5 & & & \end{array},\quad \begin{array}{llll} 1 & 1 & 1 & 5 \\ 2 & 2 & 3 & \\ 3 & 4 & 5 & \\ 4 & & & \end{array},\quad \begin{array}{llll} 1 & 1 & 1 & 5 \\ 2 & 2 & 4 & \\ 3 & 3 & 5 & \\ 4 & & & \end{array}.$$

Therefore, $S^{(4,\ 3^2,\ 1)}$ occurs with multiplicity 12.

Modular Representations

Now we understand the ordinary representations, we can now change focus to the modular representations. An advanced result within modular representation theory states that the number of irreducible modular representations is equal to the number of conjugacy classes whose elements have order coprime to $char(F) = p$. For S_n, this is equal to the number of partitions whose components are all coprime to p (as the order of an element with cycle type $(\mu_1,\ \mu_2,\ \dots, \mu_k)$ is the least common multiple of $\mu_1,\ \mu_2,\ \dots\ ,\ \mu_k$). This in turn is equal to the number of partitions of n in which no component is repeated p times, called the ***p*-regular** partitions. Otherwise we say μ is p-singular. Conveniently, it is precisely for the p-regular partitions that $D^\mu := \frac{S^\mu}{S^\mu \cap S^{\mu\perp}}$ is non-zero. This gives all the irreducible p-modular representations for S_n.

Decomposition Matrices

A *decomposition matrix* records the multiplicities of the irreducible p-modular representations D^λ (for λ p-regular) in the reductions of the irreducible ordinary representations S^μ. Explicitly the rows are parametrised by the S^μ and the columns by the D^λ, and the entry at (S^μ, D^λ) is the multiplicity of D^λ as a factor of the reduction modulo p of S^μ. The previous discussion shows, for μ p-regular, $(S^\mu,\ D^\mu)$ is non zero, indeed this factor is unique, so $(S^\mu,\ D^\mu) = 1$.

Given partitions $\mu = (\mu_1,\ \mu_2,\ \dots, \mu_k)$ and $\lambda = (\lambda_1,\ \lambda_2,\ \dots, \lambda_k)$ of n, we define a partial order by $\mu \trianglerighteq \lambda$ if and only if $\forall i,\ \sum_{j=1}^i \mu_j \geq \sum_{j=1}^i \lambda_j$. For example $(4,\ 3^2,\ 1) > (4,\ 3,\ 2^2)$.

One can show that D^μ is a factor of S^λ only if $\mu \trianglerighteq \lambda$. This forces the decomposition matrix to have the following shape:

$$\begin{pmatrix} \begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ * & & 1 \\ \hline & & \\ & * & \\ & & \end{array} \end{pmatrix}$$

Where we have ordered the columns according to the partial order and the rows such that all the p-regular partitions occur before the p-singular partitions.

As of yet, there is no general algorithm for calculating these matrices, however there are many partial results for particular characteristics and partitions. The following theorem gives a flavour of the results currently known and constructs the section of the decomposition matrix corresponding to the partitions of the form $(x,\ 1^{n-x})$, called the *hook* partitions:

Theorem: Suppose p is odd.

1. If p does not divide n, all the hook representation of S_n are irreducible as p-modular representations and no two are isomorphic.

2. If p divides n, part of the decomposition matrix of S_n is

$$\begin{array}{lccccc} (n) & 1 & 1 & & & 0 \\ (n-1,1) & & 1 & 1 & & \\ (n-2,1^2) & & & 1 & 1 & \\ \vdots & & & & \ddots & \\ (2,1^{n-2}) & & & & 1 & 1 \\ (1^n) & 0 & & & & 1 \end{array}$$

For the interested reader, I point them to "The Representation Theory of the Symmetric Groups" by G. D. James which contains thorough proofs of the information on this poster and many other results in this field.