Differential Geometry and Optimal Control

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MDO and Differential Geometry

Multidisciplinary Design Optimization (MDO) deals with the optimization of systems which have coupled subsystems; the relationships between these subsystems are defined by the state equations. We have previously developed a differential geometry framework for investigating MDO: the state equations define the feasible design space, which is a manifold (M_{feas}), and we can analyze that manifold with differential geometry.

MDO Formulation

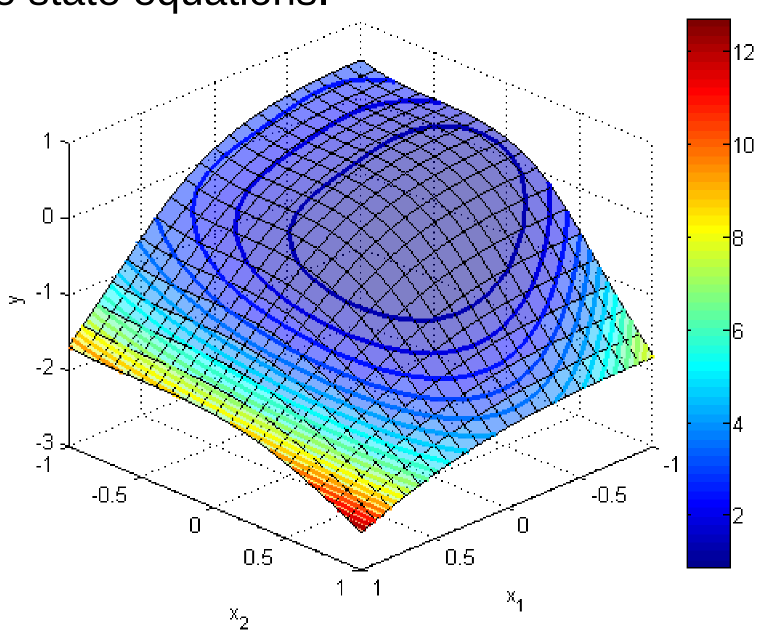
MDO problems have an objective function *f*, inequality constraints ***g***, state equations ***h***, design variables ***w***, and state variables ***y*** defined by the state equations; M_{feas} is Riemannian, and *f* is just a function on the manifold. We can calculate the (induced) metric g_{ij} on the manifold from the state equations.

$$f(\mathbf{w},\mathbf{y})$$

$$\mathbf{g}(\mathbf{w},\mathbf{y}) \le \mathbf{0}$$

$$\mathbf{h}(\mathbf{w},\mathbf{y}) = \mathbf{0}$$

$$g_{ij} = \delta_{ij} + \frac{\partial y^k}{\partial w^i}\frac{\partial y^k}{\partial w^j}$$



Project Aims and Motivations

Unlike many real designs, MDO formulations are time-invariant. We want to improve MDO by making it dynamic (i.e. incorporate optimal control). To do this, we will formulate the dynamic MDO problem, extend our framework to describe this problem with differential geometry, and do some preliminary analysis.

Dynamic MDO – Formulation

Optimal control has two parts: controller design (time-varying) and plant design (time-invariant). For dynamic MDO, the controller has objective function *S* and dynamics defined by $\boldsymbol{\xi}$, and the plant has objective *f* and state equations ***h*** (ignoring any inequalities). We now have control variables ***u***, and ***y*** changes in time. Designers typically combine *S* and *f* to make the problem single-objective.

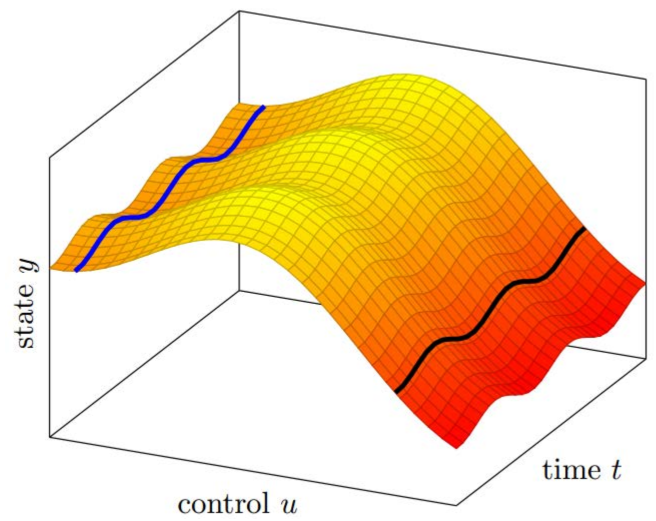
$$S(t,\mathbf{u}(t);\mathbf{w}) = \int_0^T L(t,\mathbf{u}(t),\mathbf{y}(t),\dot{\mathbf{y}}(t);\mathbf{w})\,dt \qquad f(\mathbf{w},\mathbf{u}_0,\mathbf{y}_0)$$

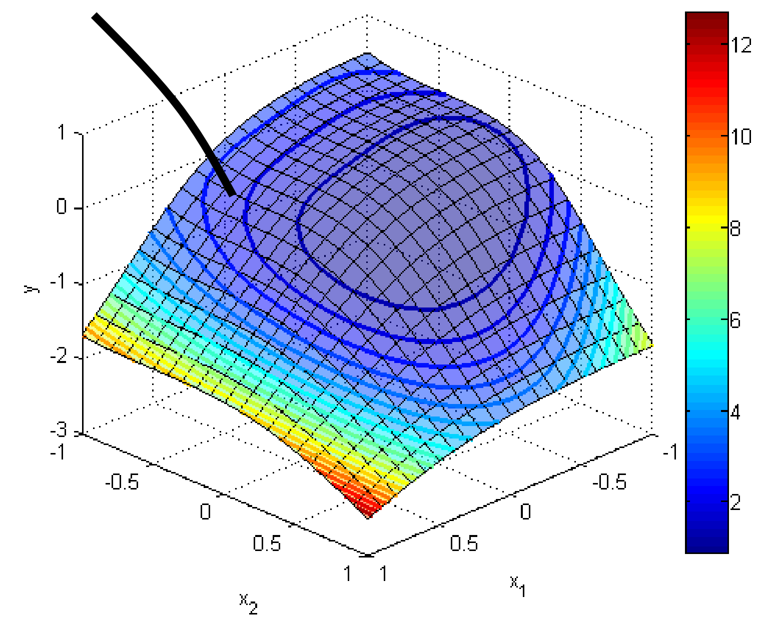
$$\dot{\mathbf{y}} = \boldsymbol{\xi}(t,\mathbf{u}(t),\mathbf{y}(t);\mathbf{w}) \qquad \mathbf{h}(\mathbf{w},\mathbf{u}_0,\mathbf{y}_0) = \mathbf{0}$$

Dynamic MDO – Manifolds and Metrics

Dynamic MDO has a fibre bundle structure: the plant is the base, and the controller is the fibre. The base is Riemannian, and depending on how time is handled, the fibre is either Riemannian or pseudo-Riemannian; there are potential analogies to relativity in the pseudo-Riemannian case. Either way, we can calculate the metrics for both the base and the fibre.

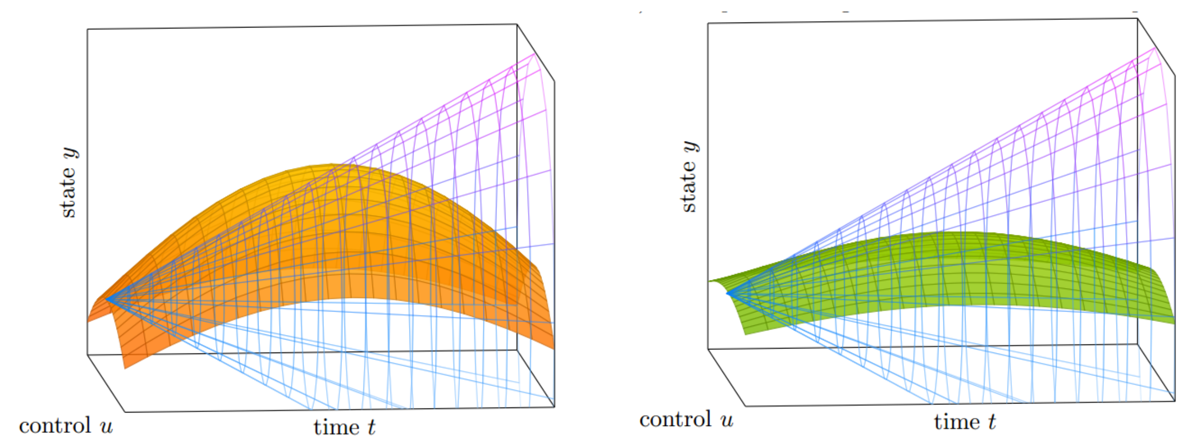
$$g_{ij} = \delta_{ij} + \frac{\partial y^k}{\partial u^i}\frac{\partial y^k}{\partial u^j} \qquad g_{ij} = \delta_{ij} + \frac{\partial y^k}{\partial(\mathbf{w},\mathbf{u}_0)^i}\frac{\partial y^k}{\partial(\mathbf{w},\mathbf{u}_0)^j}$$





Dynamic MDO – Constraints and Curvature

Constraints on variables define manifold boundaries, and constraints on variable derivatives act like light cones limiting feasible control trajectories. We can see how manifold curvature could interact with these 'light cones' to alter the range of potential trajectories.



Hamiltonian and Lagrangian Perspectives

We are currently using a Lagrangian approach – only spatial coordinates and time – but we could consider a Hamiltonian perspective, and this would produce a phase space instead. These different viewpoints could each provide different insights into dynamic MDO.

Summary and Recommendations

We developed a dynamic MDO formulation and translated it into our differential geometry framework. As a result, we can now visualize and interpret the problem's structure. With this foundation in place, we can look at open questions in dynamic MDO and use differential geometry tools to solve them.

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