Second Neighbourhood Conjecture

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This project was about a very interesting open problem

**Seymour's Second Neighbourhood Conjecture.** *Every oriented graph has a vertex v such that* $|N^2(v)| \geq |N(v)|$.

**Figure 1: First and second neighbourhood**

Let G be a simple graph. Directing each of its edges we get an oriented graph. A neighbourhood N \( pvq \) of a vertex v are all the out neighbours of v and its second neighbourhood N\( 2pvq \) consists of all the out neighbours of N \( pvq \) that are not themselves in N\( pvq \). We've proved that there are no counterexamples to the conjecture with minimal degree at most 5. We also considered stretched graphs - graphs preserving the rst neighbourhood of vertices, but presenting all the information about a graph we started with in a more clear way. The stretched graphs are eventually periodic, and a very interesting part of the project consisted of studying the period of a class of graphs.
Figure 3: Directed cycles with a common point

\[ C_5 \vee C_7 \]
Consider a graph \( C_i \sim C_j \), consisting of directed cycles \( C_i \) and \( C_j \) with a point in common. The period of stretched graphs \( C_i \sim C_j \) for small \( i, j \) is shown in figure 4. Without the colours, the table looks quite chaotic. But consider the entries \( p_i; q_j \) with \( \text{hcf} p_i; q_j \neq 2 \) (coloured yellow). Now consider the rows. There are a lot of 3's in the first row, 4's in the second row, 5's in the third one and so on. Colour those entries red. Similarly, colour the entries in the columns blue. We can see that the whole table is coloured, so the period of \( C_i \sim C_j \) is \( \text{hcf} p_i; q_j, i \) or \( j \). It turns out that we can determine whether it is \( i \) or \( j \) by considering the number of steps the Euclid's algorithm on \( i \) and \( j \) takes. The proof is in the report. References i) Seymour's second neighbourhood conjecture (http://www.math.uiuc.edu/~west/openp/2ndnbhd.html) ii) Bollobas, Bela: Modern Graph Theory. iii) Report (Insert the link to the report!)