Second Neighbourhood Conjecture

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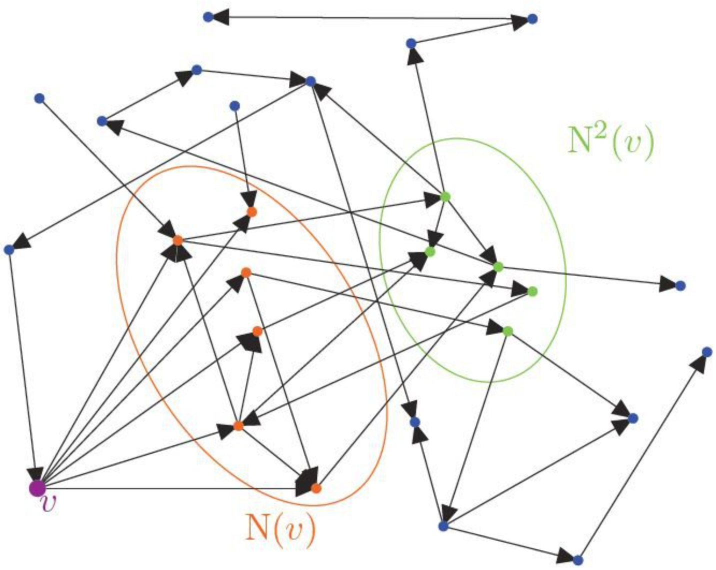
This project was about a very interesting open problem

Seymour's Second Neighbourhood Conjecture. *Every oriented graph has a vertex v such that*

$$|\mathrm{N}^2(v)| \geqslant |\mathrm{N}(v)|.$$

Figure 1: First and second neighbourhood



Let G be a simple graph. Directing each of its edges we get an oriented graph. A neighbourhood N pvq of a vertex v are all the out neighbours of v and its second neighbourhood N2pvq consists of all the out neighbours of N pvq that are not themselves in Npvq. We've proved that there are no counterexamples to the conjecture with minimal degree at most 5. We also considered stretched graphs - graphs preserving the rst neighbourhood of vertices, but presenting all the infor- mation about a graph we started with in a more clear way. The stretched graphs are eventually periodic, and a very interesting part of the project consisted of studying the period of a class of graphs.

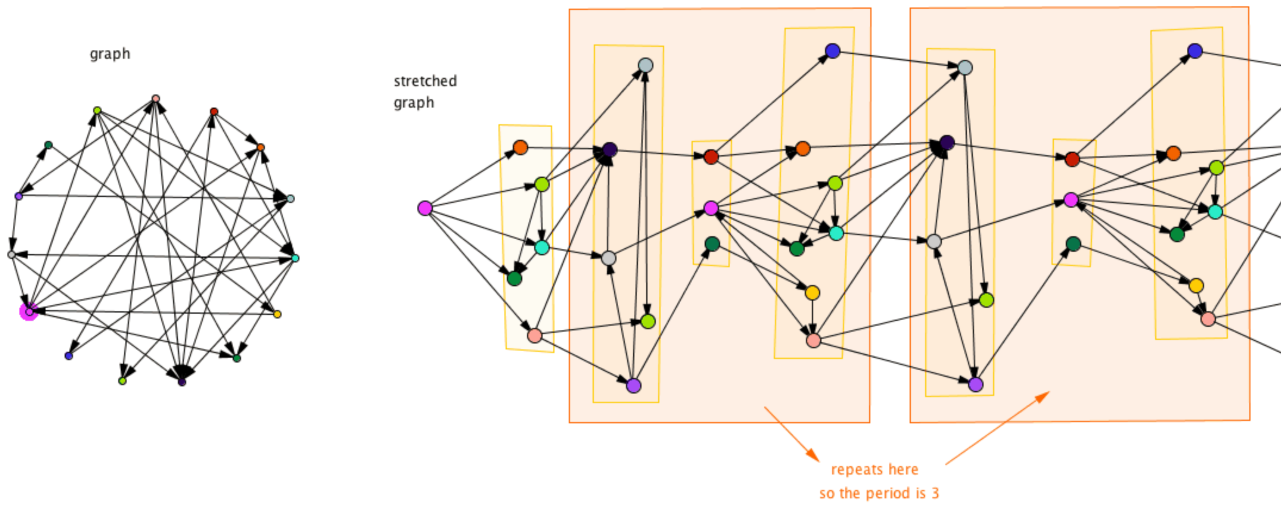


Figure 3: Directed cycles with a common point

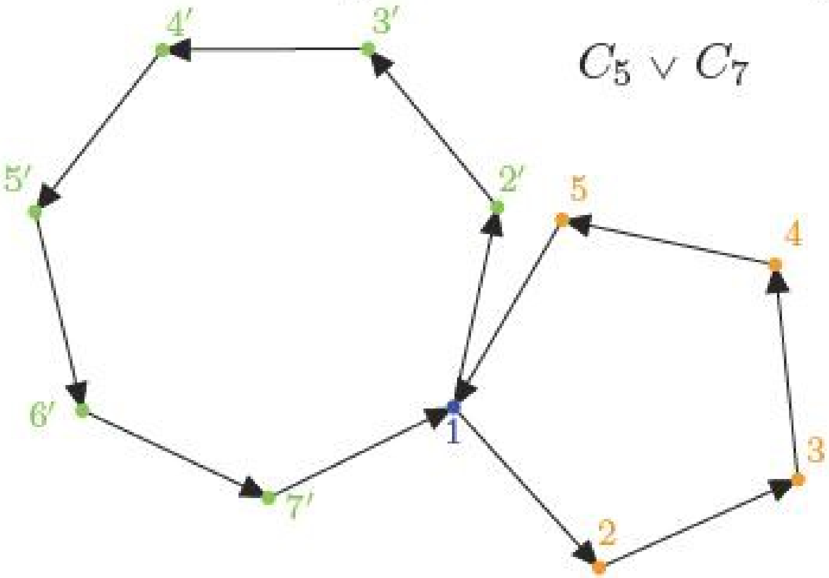


Figure 4: $[i, j]$ for small $i, j > 3$

3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
3	4	4	4	7	4	4	4	11	4	4	4	15	4	4	4	19	4	4	4	23	4	4
3	4	5	5	5	8	9	5	5	5	13	14	5	5	5	18	19	5	5	5	23	24	5
3	4	5	6	6	6	3	10	11	6	6	6	3	16	17	6	6	6	3	22	23	6	6
3	7	5	6	7	7	7	10	7	12	13	7	7	7	17	7	19	20	7	7	7	24	7
3	4	8	6	7	8	8	8	11	4	8	14	15	8	8	8	19	4	8	22	23	8	8
3	4	9	3	7	8	9	9	9	3	13	9	3	16	17	9	9	9	3	22	9	3	25
3	4	5	10	10	8	9	10	10	10	13	14	5	10	10	18	19	10	10	10	23	24	5
3	11	5	11	7	11	9	10	11	11	11	14	11	16	11	18	11	20	21	11	11	11	25
3	4	5	6	12	4	3	10	11	12	12	12	3	4	17	6	12	4	3	22	23	12	12
3	4	13	6	13	8	13	13	11	12	13	13	13	16	17	13	19	13	21	13	13	24	25
3	4	14	6	7	14	9	14	14	12	13	14	14	14	17	18	14	20	7	14	23	14	14
3	15	5	3	7	15	3	5	11	3	13	14	15	15	15	3	15	5	3	22	15	3	5
3	4	5	16	7	8	16	10	16	4	16	14	15	16	16	16	19	4	21	16	23	8	16
3	4	5	17	17	8	17	10	11	17	17	17	15	16	17	17	17	20	21	22	17	17	25
3	4	18	6	7	8	9	18	18	6	13	18	3	16	17	18	18	18	3	22	18	6	25
3	19	19	6	19	19	9	19	11	12	19	14	15	19	17	18	19	19	19	22	19	19	25
3	4	5	6	20	4	9	10	20	4	13	20	5	4	20	18	19	20	20	20	23	4	5
3	4	5	3	7	8	3	10	21	3	21	7	3	21	21	3	19	20	21	21	21	3	25
3	4	5	22	7	22	22	10	11	22	13	14	22	16	22	22	22	20	21	22	22	22	25
3	23	23	23	7	23	9	23	11	23	13	23	15	23	17	18	19	23	21	22	23	23	23
3	4	24	6	24	8	3	24	11	12	24	14	3	8	17	6	19	4	3	22	23	24	24
3	4	5	6	7	8	25	5	25	12	25	14	5	16	25	25	25	5	25	25	23	24	25

Consider a graph Ci _ Cj , consisting of directed cycles Ci and Cj with a point in common. The period of stretched graphs Ci _ Cj for small I; j is shown in gure 4. Without the colours, the table looks quite chaotic. But consider the entries pi; jq with hcfpi; jq d ¡ 2 (coloured yellow). Now consider the rows. There are a lot of 3's in the rst row, 4's in the second row, 5's in the third one and so on. Colour those entries red. Similarly, colour the entries in the columns blue. We can see that the whole table is coloured, so the period of Ci _ Cj is hcf pi; jq, I or j. It turns out that we can determine whether it is I or j by considering the number of steps the Euclid's algorithm on I and j takes. The proof is in the report. References i) Seymour's second neighbourhood conjecture (http://www.math.uiuc.edu/ west/openp/2ndnbhd.html) ii) Bollobas, Bela: Modern Graph Theory. Iii) Report (Insert the link to the report!)