# Combinatorics of KP solitons

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Photo taken in Nuevo Vallarta, Mexico by Mark J. Ablowitz

# Outline

1. The KP equation (physical motivation, mathematical structure)
2. Solitons and their contour plots
3. Combinatorial methods in the literature (two papers)
4. Our framework (via combinatorial geometry)

# The KP equation

- The Kadomtsev–Petviashvili (KP) equation is

$$\boxed{\frac{\partial}{\partial x}\left(-4\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) + 3\frac{\partial^2 u}{\partial y^2} = 0} \qquad \text{(KP equation)}$$

- $t$ time, $x, y$ spatial variables, $u$ height of the water surface
- Models shallow water waves primarily travelling in the $x$-direction
- Introduced to study stability of KdV solutions under weak transverse (i.e. $y$-direction) perturbations; cf. the KdV equation:

$$-4\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \qquad \text{(KdV equation)}$$

# Sketch of a possible "derivation"

- Restrict to 2d, so water surface $z = u(t, x)$. Make simplifying assumptions (no dispersion, linear, shallow, etc.) to obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}, \quad \text{solved by e.g. } u = e^{i(kx-\omega(k)t)} \text{ with } \omega(k) = vk$$

- Reintroduce *weak* dispersion (e.g. try $\omega(k) = v(k - \beta k^3)$ and see what equation $u$ satisfies) and *weak* nonlinearity:

$$\text{1d wave equation} \xrightarrow{\text{weak dispersion + nonlinearity}} \text{KdV equation}$$

- Introduce weak variation in a transverse direction:

$$\text{KdV equation} \xrightarrow{\text{weak variation in the } y\text{-direction}} \text{KP equation}$$

# The KP equation is an integrable system

- Infinitely many conserved quantities "in involution"
- ($\xrightarrow{\text{Noether}}$) Infinitely many "generalized symmetries"
- Gives infinite family of compatible PDEs called **KP hierarchy**:

$$\frac{\partial u}{\partial t_N} = \text{expression in } u \text{ and its derivatives w.r.t. } t_1, t_2, \dots, t_{N-1}$$

- KP hierarchy admits a large class of exact solutions called **solitons**:

$$\boxed{u = u(t_1, t_2, t_3, t_4, t_5, \dots)}$$

- Identify $t_1 = x$, $t_2 = y$, $t_3 = t$ and assume **higher times** $t_4, t_5, \dots$ eventually zero; write $\mathbf{t} = (t_1, \dots, t_m)$

# Recipe to construct a soliton

- Fix $\boxed{n, k \in \mathbb{N}_0,}$ with $k \leq n$
- Pick $n$ real constants $\kappa_1 < \kappa_2 < \cdots < \kappa_n$ and define linear forms $\theta_j(\mathbf{t}) = \sum_{l=1}^{m} \kappa_j^l t_l$ for $j = 1, \ldots, n$
- Pick a full-rank $k \times n$ matrix $A$
- Denote $[n] := \{1, \ldots, n\}$ and $\binom{[n]}{k} := \{\text{size } k \text{ subsets of } [n]\}$
- For each $J \in \binom{[n]}{k}$, pick out the $k$ columns of $A$ indexed by $J$; the determinant is $\Delta_J(A)$; ensure $\Delta_J(A) \geq 0$ ($\Delta_J(A)$ are the **Plücker coordinates** of the point $A$ on the **Grassmannian** $\mathrm{Gr}(n, k)$)
- Soliton given by

$$u_A = 2\frac{\partial^2}{\partial x^2} \log \tau_A, \quad \text{where } \tau_A = \sum_{J \in \binom{[n]}{k}} \Delta_J(A) K_J \exp\left[\sum_{j \in J} \theta_j(\mathbf{t})\right]$$

where $K_J > 0$ depends on $J$ and the $\kappa_j$'s

# "Tropical" approximation

- At "most" points, have one dominant term in the sum, so

$$\log \tau_A \approx \max_{J \in \binom{[n]}{k}} \left\{ \sum_{j \in J} \theta_j(\mathbf{t}) + \log(\Delta_J(A) K_J) \right\}$$

- Problematic if $\Delta_J(A) = 0$, so really should max over

$$\boxed{\mathcal{J} := \left\{ J \in \binom{[n]}{k} : \Delta_J(A) > 0 \right\} \subset \binom{[n]}{k}}$$

- Further assume scale of $\mathbf{t}$ is large, so discard constant terms entirely:

$$\log \tau_A \approx \max_{J \in \mathcal{J}} \left\{ \sum_{j \in J} \theta_j(\mathbf{t}) \right\} =: f_{\mathcal{J}}(\mathbf{t}) \qquad \text{(Tropical approximation)}$$

# Aside: tropical geometry

- Often described as a "piecewise-linear version of algebraic geometry"
- Replace $+$ and $\times$ with **tropical addition** and **multiplication**:

$$x \oplus y = \max\{x, y\}, \quad x \otimes y = x + y \quad \text{for } x, y \in \mathbb{R} \cup \{-\infty\}$$

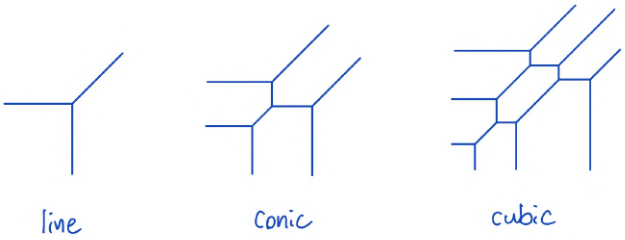
  have $\oplus, \otimes$ associative and commutative, $\otimes$ distributes over $\oplus$, etc.
- **Tropical polynomials**: e.g. $4x^3 + xy + y^2$ becomes

$$(4 \otimes x^{\otimes 3}) \oplus (x \otimes y) \oplus (y^{\otimes 2}) = \max\{3x + 4, x + y, 2y\},$$

  interpret as a piecewise-linear function $f : \mathbb{R}^2 \to \mathbb{R}$

# Aside: tropical geometry

- Instead of zero sets (AG), we care about points where the tropical polynomial is nonlinear, i.e. **maximum achieved at least twice;** this set of points is called the **tropical variety** of $f$, written $V(f)$




- Many classical AG theorems have tropical analogs: Bézout's theorem, genus-degree formula, Riemann–Roch, etc.

# Contour plot = tropical variety of $f_{\mathcal{J}}$

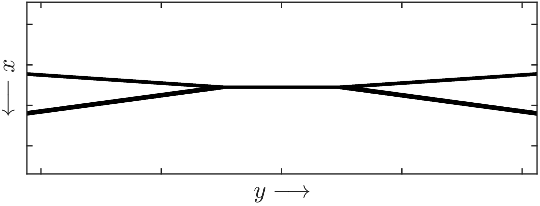
- Goal: visualize $u_A$ in $\mathbb{R}^2$. So treat $(t_3, \ldots, t_m)$ as constants, get linear functions $\theta_j : \mathbb{R}^2 \to \mathbb{R}$
- Given the context of tropical geometry, we call

$$f_{\mathcal{J}} = \max_{J \in \mathcal{J}} \left\{ \sum_{j \in J} \theta_j \right\} : \mathbb{R}^2 \to \mathbb{R} \quad \text{a } \textbf{tropical function}$$

- $u_A = 2\partial_x^2 \log \tau_A \approx 2\partial_x^2 f_{\mathcal{J}}$, so $u_A \approx 0$ UNLESS the point lies in the **tropical variety** $V(f_{\mathcal{J}})$; call $V(f_{\mathcal{J}}) \subset \mathbb{R}^2$ the **contour plot** of $u_A$
- (Locations of) wave crests of $u_A$ are approximated by $V(f_{\mathcal{J}})$
- More generally, fix a dimension $d \geq 3$ and treat $(t_d, \ldots, t_m)$ as constants, so have $f_{\mathcal{J}} : \mathbb{R}^{d-1} \to \mathbb{R}$ and $V(f_{\mathcal{J}}) \subset \mathbb{R}^{d-1}$

# Contour plots in $d = 3$ vs. photographs

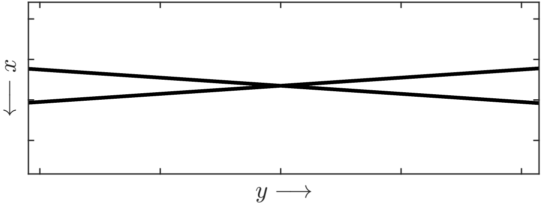
$n = 4$, $k = 2$, $\mathcal{J} = \binom{[4]}{2} \setminus \{34\}$
$(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (-0.3, -0.1, 0.1, 0.3)$, $t = 10$




Taken on Venice Beach, California by Douglas Baldwin.

# Contour plots in $d = 3$ vs. photographs

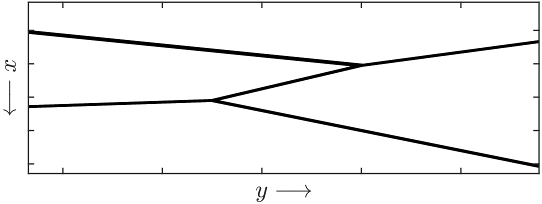
$n = 4$, $k = 2$, $\mathcal{J} = \binom{[4]}{2} \setminus \{13\}$
$(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (-0.3, -0.1, 0.1, 0.3)$, $t = 10$




Taken in Nuevo Vallarta, Mexico by Mark Ablowitz.

# Contour plots in $d = 3$ vs. photographs

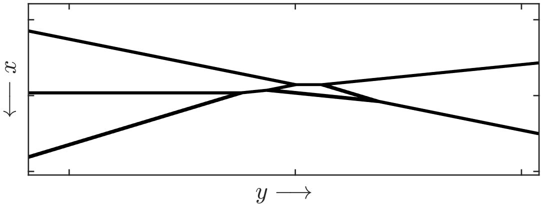
$n = 4$, $k = 2$, $\mathcal{J} = \binom{[4]}{2} \setminus \{23\}$
$(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (-0.6, -0.3, 0, 0.7)$, $t = -10$




Taken on Venice Beach, California by Douglas Baldwin.

# Contour plots in $d = 3$ vs. photographs

$n = 5$, $k = 2$, $\mathcal{J} = \binom{[5]}{2} \setminus \{45\}$
$(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5) = (-0.6, -0.3, 0, 0.3, 0.6)$, $t = 2$

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# Summary of important parameters

Recap: $u_A = 2\partial_x^2 \log \tau_A$ soliton, $f_{\mathcal{J}} \approx \log \tau_A$ tropical approximation, $V(f_{\mathcal{J}})$ contour plot approximating crest locations

- $k$: number of rows of $A$
- $n$: number of columns of $A$
- $d$: dimension of space in which the contour plot lives $(+1)$
- $\mathcal{J}$: size $k$ index subsets $J \subset \{1, \ldots, n\}$ such that $\Delta_J(A) > 0$

# Work of Dimakis and Müller-Hoissen (DM)

- Dimakis and Müller-Hoissen (2012) classified contour plots when:

$$\boxed{k = 1,\ \mathcal{J} = [n], \text{ and } n, d \text{ arbitrary}}$$

- They did this using combinatorial objects (specifically posets) called **higher Bruhat and Tamari orders**, for example:

**Definition (Higher Bruhat order)**

Let $L = \{i_1, \ldots, i_{d+2}\} \in \binom{[n]}{d+2}$ and $P(L)$ its packet. A *beginning segment* is a subset of $P(L)$ of the form $\{L - i_{d+2}, L - i_{d+1}, \ldots L - i_j\}$ for some $j$. An *ending segment* is a subset of the form $\{L - i_j, L - i_{j-1}, \ldots L - i_1\}$ for some $j$. A subset $\mathcal{K} \subset \binom{[n]}{d+1}$ is *consistent* if $\mathcal{K} \cap P(L)$ is a beginning or ending segment for any $L \in \binom{[n]}{d+2}$.
The *higher Bruhat order* $B(n, d)$ is the set of consistent subsets of $\binom{[n]}{d+1}$, ordered by single step inclusion.

# Work of Kodama and Williams (KW)

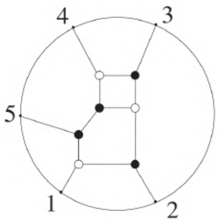
- Kodama and Williams (2014) investigated contour plots when:

$$\boxed{d = 3 \text{ and } n, k, \mathcal{J} \text{ arbitrary}}$$

- They did this using **plabic graphs** (among *many* other things)

**Definition (Plabic graph)**

A *plabic graph* is a planar undirected graph $G$ drawn inside a disk with $n$ *boundary vertices* $1, \ldots, n$ placed in counterclockwise order around the boundary of the disk, such that each boundary vertex $i$ is incident to a single edge. Each internal vertex is colored black or white.



# Idea of the project

- Higher Bruhat orders admit a natural geometric interpretation in terms of **zonotopal tilings** of **cyclic zonotopes**:

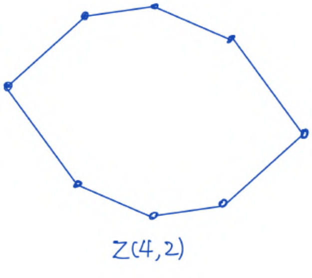
$$B(n, d) \longleftrightarrow \{\text{Zonotopal tilings of } Z(n, d)\}$$

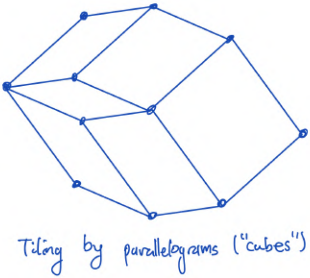
- Furthermore, Galashin (2018) showed that plabic graphs are equivalent to **cross sections** of zonotopal tilings of $Z(n, 3)$

$\therefore$ consolidate both approaches by studying zonotopes directly

# Zonotopes and zonotopal tilings

- A **zonotope** is the image of an $n$-cube $[0,1]^n$ under a linear map $X : \mathbb{R}^n \to \mathbb{R}^d$, denoted as $Z(X) = X([0,1]^n) \subset \mathbb{R}^d$
- A zonotope is **cyclic** if all minors of $X$ are positive; write $Z(X) = Z(n,d)$ in this case
- $Z(d,d)$ is called a **cube** (but they're really $d$-parallelepipeds)
- A **(fine) zonotopal tiling** of $Z(n,d)$ is a subdivision into cubes



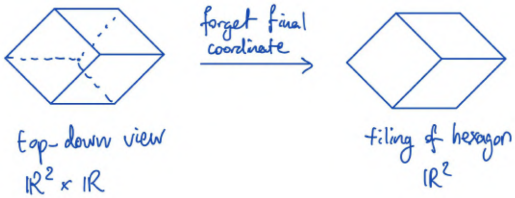



# Regular tilings

- Zonotope $Z(X) \subset \mathbb{R}^d$, where $X$ is a $d \times n$ matrix
- Assign "weights" $c_1, \ldots, c_n$ to the column vectors of $X$; get $(d+1) \times n$ matrix

$$\widehat{X} = \begin{pmatrix} X \\ \hline c_1 \quad \ldots \quad c_n \end{pmatrix}$$

- The upper "facets" of $Z(\widehat{X}) \subset \mathbb{R}^d \times \mathbb{R}$ project to a tiling of $Z(X)$
- E.g. upper faces of a cube projects to a tiling of a hexagon



# The main theorem

- $\mathcal{J} = \binom{[n]}{k}$ from now on ("totally positive" case, otherwise need **matroid polytopes**); recall:

$$f_{\binom{[n]}{k}} = \max_{J \in \binom{[n]}{k}} \{\textstyle\sum_{j \in J} \theta_j\}, \quad \text{where } \theta_j = \textstyle\sum_{l=1}^{d-1} \kappa_j^l t_l + c_j$$

- Shorthand: $V_k = V\left(f_{\binom{[n]}{k}}\right)$, "$k^{\text{th}}$ variety/contour plot"

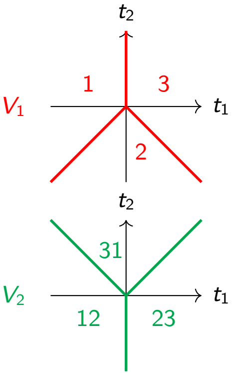
**Theorem**

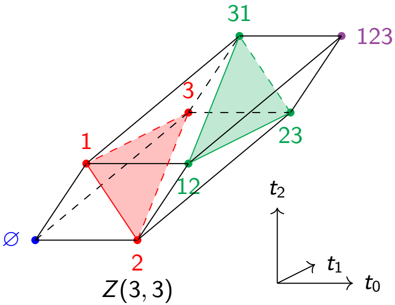
*$V_k \subset \mathbb{R}^{d-1}$ is the $(d-2)$-skeleton of the polyhedral complex dual to the $k^{th}$ cross section of the regular tiling of the cyclic zonotope $Z(n,d) \subset \mathbb{R} \times \mathbb{R}^{d-1}$ generated by the data*

$$\widehat{X} = \left(\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ \kappa_1 & \kappa_2 & \cdots & \kappa_n \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_1^{d-1} & \kappa_2^{d-1} & \cdots & \kappa_n^{d-1} \\ \hline c_1 & c_2 & \cdots & c_n \end{array}\right)$$

# A toy example

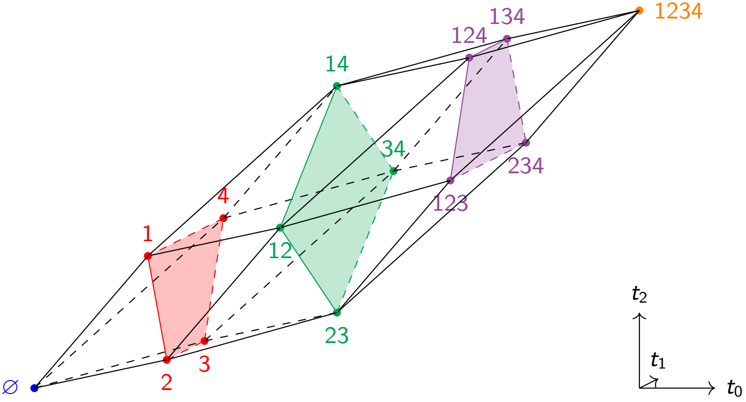
$d = 3$, $n = 3$, $(\kappa_1, \kappa_2, \kappa_3) = (-1, 0, 1)$, and $t = 0$
Affine forms $\theta_1 = -t_1 + t_2$, $\theta_2 = 0$, $\theta_3 = t_1 + t_2$



# A better example

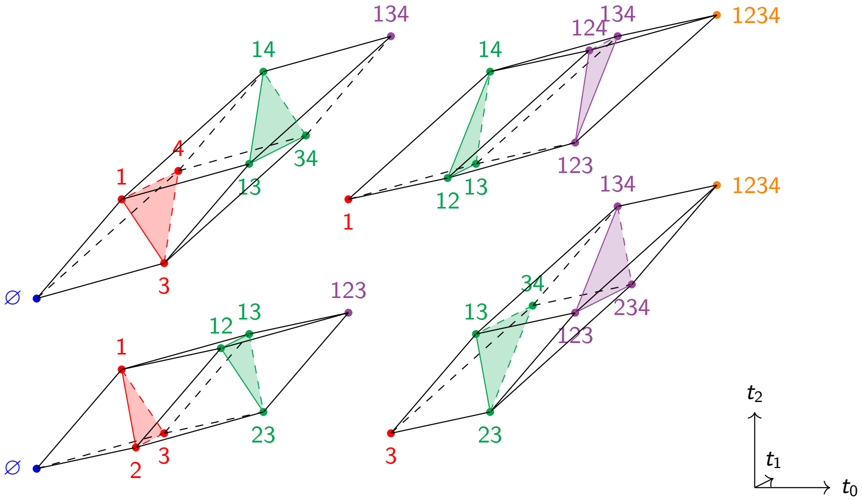
$d = 3$, $n = 4$, $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (-1, -0.5, 0.5, 1)$, $t = -1$
Affine forms $\theta_1, \theta_2, \theta_3, \theta_4$ (I won't write them out)



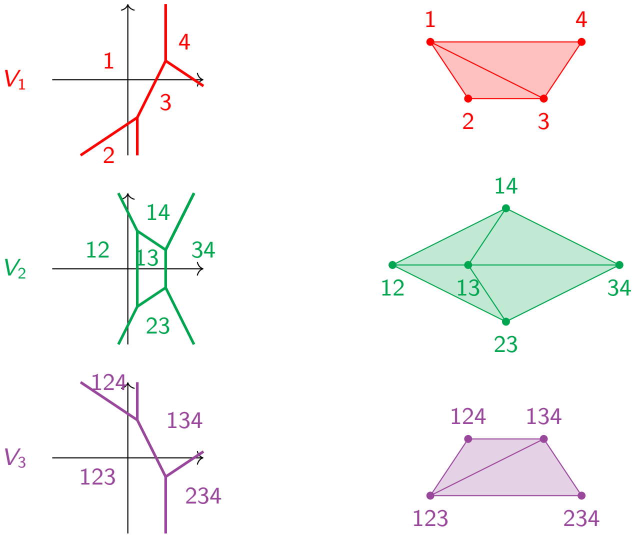

$Z(4, 3)$

# A better example

$t = -1$ gives this tiling:



# A better example



# Parameters in the dual picture

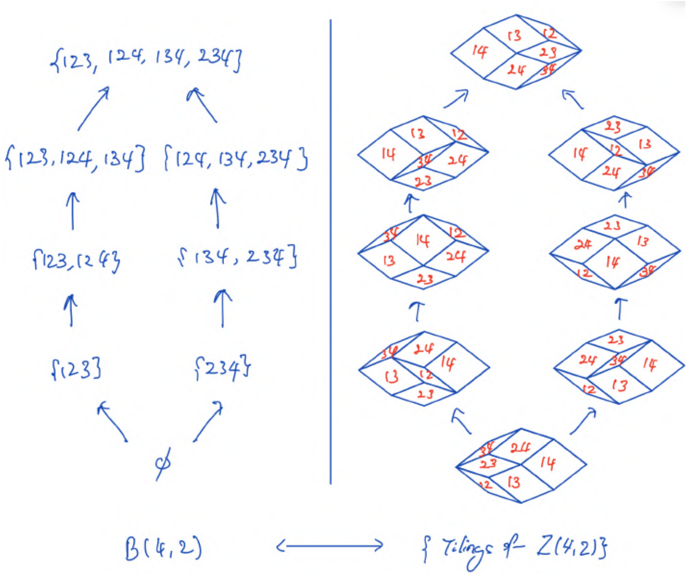
Recall the original interpretations:

- $k$: number of rows of $A$
- $n$: number of columns of $A$
- $d$: dimension of space in which the contour plot lives $(+1)$
- $\mathcal{J}$: size $k$ index subsets $J \subset \{1, \ldots, n\}$ such that $\Delta_J(A) > 0$

In the **dual (geometric) picture**:

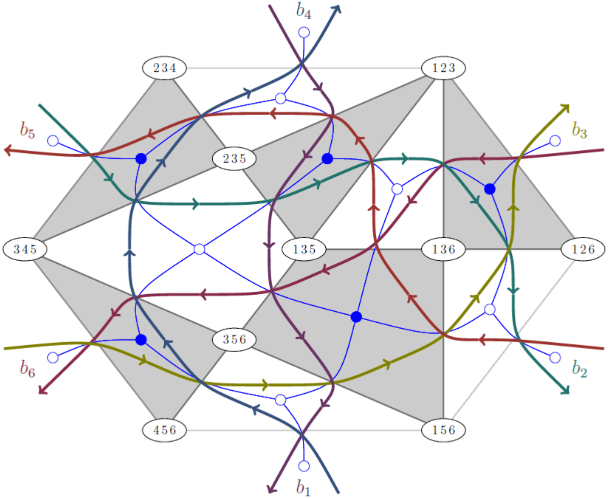
- $k$: cross section level in $Z(n, d)$
- $n$: number of generating vectors of $Z(n, d)$
- $d$: dimension of $Z(n, d)$
- $\mathcal{J}$: vertices of matroid polytope $\subset \Delta^{n,k}$

# Relation to higher Bruhat orders (DM)



# Relation to plabic graphs (KW)

From Galashin's 2018 paper:



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