Research Summary: *Combinatorics of KP solitons*

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In this project, we unified different combinatorial approaches to KP solitons due to Dimakis–Müller-Hoissen and Kodama–Williams. To this end, we established a duality between tropical varieties approximating the solitons and cross sections of zonotopal tilings of cyclic zonotopes.

1. Background

The *Kadomtsev–Peviashvili (KP) equation* is a nonlinear dispersive wave equation modelling shallow water waves primarily traveling in the x-direction, given by

$$\partial_x(-4\partial_t u + 6u\partial_x u + \partial_x^3 u) + 3\partial_y^2 u = 0.$$

It is a prototypical *integrable system*, admitting infinitely many conserved quantities in involution, leading to an infinite system of compatible PDEs called the *KP hierarchy*. A solution to this system has the form $u = u(t_1 = x, t_2 = y, t_3 = t, t_4, \dots)$. Assume that these *higher times* are eventually zero, say after t_m, and write $\mathbf{t} = (t_1, \dots, t_m)$.

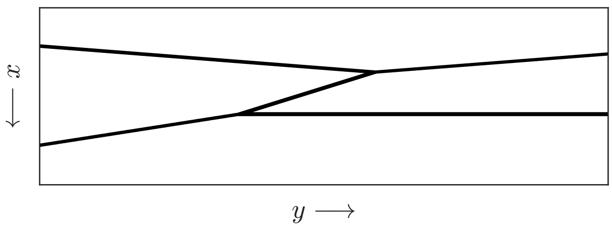
The KP hierarchy admits a large class of exact solutions called *solitons*. To construct one, fix $n, k \in \mathbb{N}$, $k \le n$, real constants $\kappa_1 < \kappa_2 < \cdots < \kappa_n$, and a $k \times n$ matrix A with *Plücker coordinates* $\Delta_J(A) \ge 0$, $J \in \binom{[n]}{k} = \{\text{size } k \text{ subsets of } [n] = \{1, \dots, n\}\}$. The soliton is given by

$$u_A(\mathbf{t}) = 2\partial_x^2 \log\left[\textstyle\sum_{J\in\binom{[n]}{k}} \Delta_J(A) K_J \exp(\sum_{j\in J}\theta_j(\mathbf{t}))\right],$$

where $K_J > 0$ is constant and $\theta_j(\mathbf{t}) = \sum_{l=1}^m \kappa_j^l t_l$, $j \in [n]$. At large scales of $\mathbf{t}$, we have the approximation

$$\log[\dots] \approx \max_{J\in\mathcal{J}}\{\textstyle\sum_{j\in J}\theta_j(\mathbf{t})\} = f_{\mathcal{J}}(\mathbf{t}),$$

where $\mathcal{J} \subset \binom{[n]}{k}$ is the subset of J's for which $\Delta_J(A) > 0$. Fix a "dimension" d and view the times $(t_d, \dots, t_m)$ as constants. Since $u_A \approx 0$ when $f_{\mathcal{J}}$ is linear, we are interested in the set of points $V(f_{\mathcal{J}}) \subset \mathbb{R}^{d-1}$ where $f_{\mathcal{J}}$ is not linear, i.e. *the maximum over $\mathcal{J}$ is achieved at least twice.* We call $V(f_{\mathcal{J}})$ the *tropical variety* of $f_{\mathcal{J}}$ (from the fascinating field of tropical geometry, in some sense a piecewise-linear version of algebraic geometry). We also call $V(f_{\mathcal{J}})$ the *asymptotic contour plot* of u_A, which in $d = 3$ approximates (the locations of) the wave crests of u_A.



(a) An asymptotic contour plot in $d = 3$ with $n = 4$, $k = 2$.

(b) Taken on Venice Beach, California by Douglas Baldwin [1].

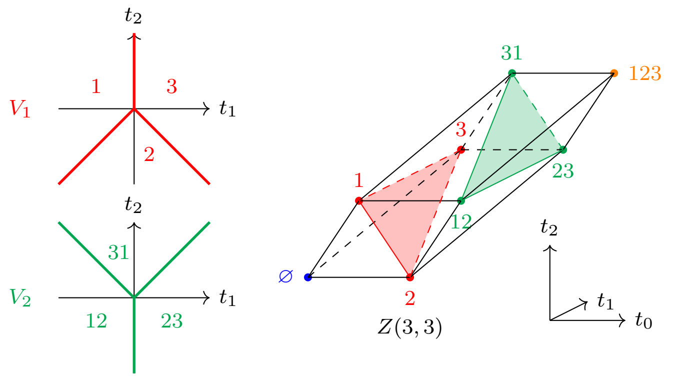
The "discrete" nature of contour plots opens them up to combinatorial study. Dimakis and Müller-Hoissen used objects called *higher Bruhat and Tamari orders* to classify contour plots with $k = 1$, $\mathcal{J} = [n]$, and arbitrary n, d [2]. Meanwhile, Kodama and Williams used *(generalized) plabic graphs* and various combinatorial objects describing the *totally non-negative Grassmannian* to investigate contour plots in $d = 3$ with arbitrary $n, k, \mathcal{J}$ [3].

2. Our framework

The idea of the project is the unify these two approaches using dual objects from combinatorial geometry called *zonotopal tilings* of *cyclic zonotopes* $Z(n, d)$. For example, elements of the higher Bruhat order $B(n, d)$ have a natural interpretation as tilings of $Z(n, d)$. Furthermore, it was shown in [4] that plabic graphs are equivalent to cross sections of tilings of $Z(n, 3)$, but these do not generalize to higher d. The following is the main theorem establishing our proposed duality when $\mathcal{J} = \binom{[n]}{k}$ (if $\mathcal{J} \ne \binom{[n]}{k}$, we need to use objects called *matroid polytopes*).

Theorem. $V_k = V(f_{\binom{[n]}{k}}) \subset \mathbb{R}^{d-1}$ *is the $(d-2)$-skeleton of the polyhedral complex dual to the k^{th} cross section of the regular zonotopal tiling of the cyclic zonotope $Z(n, d) \subset \mathbb{R} \times \mathbb{R}^{d-1}$ generated by the affine forms $t_0 + \theta_j(t_1, \dots, t_{d-1})$.*

Example. A simple toy example will help illustrate this. Let $d = 3$, $n = 3$, $(\kappa_1, \kappa_2, \kappa_3) = (-1, 0, 1)$, and $t_3 = t_4 = \cdots = 0$. This gives $\theta_1 = -t_1 + t_2$, $\theta_2 = 0$, and $\theta_3 = t_1 + t_2$. Shown are V_k for $k = 1$ and $k = 2$, and the zonotope $Z(3, 3)$:



Using our framework, we aim to (1) classify contour plots for arbitrary $n, k, d, \mathcal{J}$ in the limit $t_d \to \pm\infty$, and (2) algorithmically construct contour plots for arbitrary $n, k, \mathbf{t}$ in $d = 3$, $\mathcal{J} = \binom{[n]}{k}$. A preprint is expected in 2026.

References

[1] D. Baldwin, "Nonlinear Waves." `https://douglasbaldwin.com/nl-waves.html`. Accessed: 2025/09/15.

[2] A. Dimakis and F. Müller-Hoissen, "KP solitons, higher Bruhat and Tamari orders," *Associahedra, Tamari Lattices and Related Structures*, p. 391–423, 2012.

[3] Y. Kodama and L. Williams, "KP solitons and total positivity for the Grassmannian," *Invent. math.*, vol. 198, pp. 637–699, 2014.

[4] P. Galashin, "Plabic graphs and zonotopal tilings," *Proc. Lond. Math. Soc.*, vol. 117, pp. 661–681, 2018.