Residual Finiteness of Graph Wreath Products

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The Isomorphism Problem

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- We may want partial algorithms
- One such algorithm is comparing finite quotients
- When does this fail?

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- This gives an instance of failure
- This motivates the study of residual finiteness

Residually Finite Groups

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Examples

- $\mathbb{Q}$ is not residually finite
- Every divisible group is not residually finite

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Theorem

Free products of residually finite groups are residually finite

Graph Products

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Given a graph $G = (V, E)$ and a group Γ, the Graph Product, $G(\Gamma)$, is the free product of Γ with itself V-many times, quotient by "two copies of Γ commute if there is an edge between their vertices"

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Theorem

The graph product of a residually finite group is residually finite (Green, 1990)

Semidirect Products

Definition

Suppose G, H are groups, and we have a homomorphism $\phi\colon H \to \mathrm{Aut}(G)$. The semi-direct product, $G \rtimes H$ is the group with underlying set $G \times H$, and operation $(g, h)(g', h') = (g\phi(h)(g'), hh')$

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Examples

- $G \rtimes H = G \times H$ if ϕ is the trivial map
- $S_3 = C_3 \rtimes C_2$
- If $N \trianglelefteq G$, $H \leq G$, $H \cap N = \{e\}$ and $HN = G$, then $G = N \rtimes H$

Graph Wreath Products

Definition

Given a graph G, groups Γ, Δ, and an action $\Gamma \circlearrowleft G$ via graph automorphisms, we define the graph wreath product of Γ and Δ over G to be $G(\Delta) \rtimes \Gamma$, with the action $\Gamma \circlearrowleft G(\Delta)$ induced by the action $\Gamma \circlearrowleft G$

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Examples

- If $G = (V, V^2)$, this is the combinatorial wreath product of Δ and Γ
- If $G = (\Gamma, \emptyset)$, then $G(\Delta) \rtimes \Gamma \cong \Gamma * \Delta$

Residual Finiteness of Wreath Products

Theorem

(Cornulier, 2014) If X is a set, $\Gamma \circlearrowright X$ and $G = (X, X^2)$, then $G(\Delta) \rtimes \Gamma$ is residually finite if and only if Δ, Γ are residually finite and either:

1. B is abelian and all vertex stabilisers are profinitely closed in G, or
2. all vertex stabilisers have finite index in G.

The special case of $X = \Gamma$ with Γ acting by left multiplication is due to (Grünberg, 1957)

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These results motivate the study of residual finiteness of graph wreath products

Residual Finiteness of Graph Wreath Products

The Main Theorem

(Needham, 2025) For groups Γ, Δ a graph $G = (V, E)$ on which Γ acts, $G(\Delta) \rtimes \Gamma$ is residually finite if and only if

1. Γ, Δ are residually finite;
2. Either Δ is abelian and for all neighbouring $v, w \in V$ there is a finite index subgroup $K \leq \Gamma$ such that $w \notin Kv$, or for all v there is some finite index subgroup $K \leq \Gamma$ such that $Kv \cap N(v) = \emptyset$;
3. for all $v, w \in X$ not neighbouring and not equal there is a finite index $K \leq \Gamma$ such that $Kw \cap (N(v) \cup \{v\}) = \emptyset$.

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Remark

The forwards direction is largely unenlightening

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Remark

So it suffices to reduce the general case to the finite case

Reduction to the Finite Graph Case

We fix a non-trivial element $(w, g) \in G(\Delta) \rtimes \Gamma$. We assume $w \neq e$

Proposition

For any normal subgroup K of $G(\Delta) \rtimes \Gamma$, there is a homomorphism $G(\Delta) \rtimes \Gamma \to (K \setminus G)(\Delta) \rtimes (\Gamma / K)$ which maps w to something non-trivial if its support is an induced subgraph of $K \setminus G$

Proposition

Conditions (2) and (3) of the main theorem ensure we may choose K so that any finite induced subgraph of G is an induced subgraph of $K \setminus G$