Hilbert distortion of the stochastic diamond graph

Ivan Vojvodić

supervised by Jason Miller

13 October 2025

Outline

Expansion and contraction

Expansion of an embedding

Given metric spaces $(X, d_X), (Y, d_Y)$ and an embedding (injective function) $f\colon X \hookrightarrow Y$, the *expansion of f* is defined as

$$e(f) := \sup_{u,v \in X, u \neq v} \frac{d_Y(f(u), f(v))}{d_X(u, v)}$$

Expansion and contraction

Expansion of an embedding

Given metric spaces $(X, d_X), (Y, d_Y)$ and an embedding (injective function) $f\colon X \hookrightarrow Y$, the *expansion of f* is defined as

$$e(f) := \sup_{u,v \in X, u \neq v} \frac{d_Y(f(u), f(v))}{d_X(u, v)}$$

Contraction of an embedding

Given metric spaces $(X, d_X), (Y, d_Y)$ and an embedding $f\colon X \hookrightarrow Y$, the *contraction of f* is defined as

$$c(f) := \sup_{u,v \in X, u \neq v} \frac{d_X(u, v)}{d_Y(f(u), f(v))}$$

Distortion

Distortion of an embedding

Given metric spaces $(X, d_X), (Y, d_Y)$ and an embedding $f: X \hookrightarrow Y$, the *distortion of f* is defined as

$$D(f) = c(f) \cdot e(f)$$

Distortion

Distortion of an embedding

Given metric spaces $(X, d_X), (Y, d_Y)$ and an embedding $f : X \hookrightarrow Y$, the *distortion of f* is defined as

$$D(f) = c(f) \cdot e(f)$$

Distortion

Given metric spaces $(X, d_X), (Y, d_Y)$ the *distortion of X in Y* is defined as

$$c_Y(X) := \inf_{f : X \hookrightarrow Y} D(f)$$

Distortion

Distortion of an embedding

Given metric spaces $(X, d_X), (Y, d_Y)$ and an embedding $f : X \hookrightarrow Y$, the *distortion of f* is defined as

$$D(f) = c(f) \cdot e(f)$$

Distortion

Given metric spaces $(X, d_X), (Y, d_Y)$ the *distortion of X in Y* is defined as

$$c_Y(X) := \inf_{f : X \hookrightarrow Y} D(f)$$

- When $Y = \ell_2$, we write $c_2(X)$ for $c_{\ell_2}(X)$.

Rao's Theorem

Graph as a metric space

Any (weighted) graph G can be viewed as a metric space using the shortest path metric.

Rao's Theorem

Graph as a metric space

Any (weighted) graph G can be viewed as a metric space using the shortest path metric.

Theorem 1 (Rao)

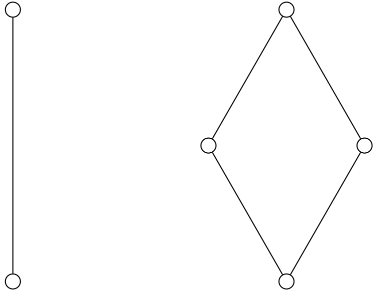
Any (weighted) planar graph G on n vertices embeds into Euclidean space with $O\left(\sqrt{\log n}\right)$ distortion. Using the notation from before,

$$c_2(G) = O\left(\sqrt{\log n}\right)$$

The Diamond Graph D_k

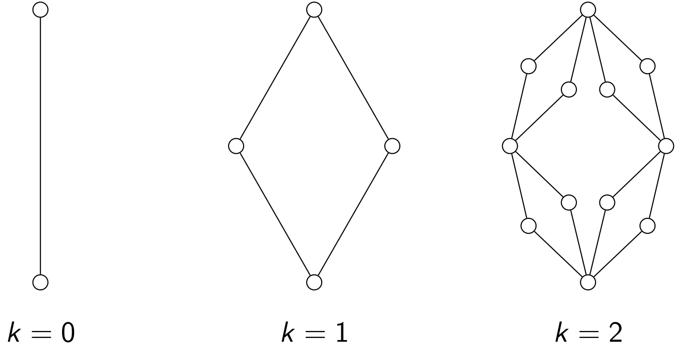
$k = 0$

The Diamond Graph D_k



$k = 0$ $\quad$ $k = 1$

The Diamond Graph D_k



The Diamond Graph D_k

Theorem 2 (Newman and Rabinovich)

The Diamond graph D_k embeds in Euclidean space with $\Omega\left(\sqrt{k}\right)$ distortion.

The Diamond Graph D_k

Theorem 2 (Newman and Rabinovich)

The Diamond graph D_k embeds in Euclidean space with $\Omega\left(\sqrt{k}\right)$ distortion.

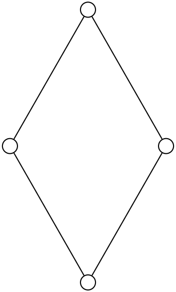
- Since $n = |V(D_k)|$ is exponential in k, this attains Rao's upper bound.

The Stochastic Diamond Graph $D_{k,p}$

$k = 0$

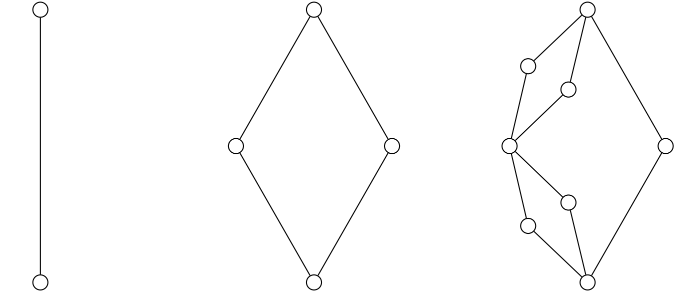
The Stochastic Diamond Graph $D_{k,p}$

$k = 0$



$k = 1$

The Stochastic Diamond Graph $D_{k,p}$



$k = 0$ $k = 1$ $k = 2$

The main question

Question

For a fixed $p \in (0, 1)$, how does the distortion $c_2(D_{k,p})$ grow as $k \to \infty$?

The main question

Question

For a fixed $p \in (0,1)$, how does the distortion $c_2(D_{k,p})$ grow as $k \to \infty$?

- For $p \leq 1/4$, the graph is finite almost surely, and thus embeds with $O(1)$ distortion.

An easy lower bound

- One can show that $D_{k,p}$ will contain a copy of $D_{\Omega(\log k)}$ with a fixed positive probability.

An easy lower bound

- One can show that $D_{k,p}$ will contain a copy of $D_{\Omega(\log k)}$ with a fixed positive probability.
- This shows that $c_2(D_{k,p}) = \Omega\left(\sqrt{\log k}\right)$, conditioned on nonextinction.

Main result

Theorem

Let $p_c = \sqrt{27/32}$. Conditioned on nonextinction, we have the following:

- If $1/4 < p < p_c$, then the distortion grows as $\sqrt{\log k}$.
- If $p_c < p < 1$, then the distortion grows as $\sqrt{k}$.

Thank you!