Maximal Singularities on the Hilbert Scheme of points

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Main results

Conjecture [Briançon-Iarrobino 1978]

Briançon and Iarrobino made a prediction for where the worst point on $Hilb^l \mathbb{A}^N$ is.

Theorem [Mackenzie-Rezaee 2025]

The above Conjecture is true for $N = 3$.

Intro to Hilbert Scheme

- The idea of a Hilbert Scheme is to parametrise the subschemes of a variety.
- It has some nice properties: if we start with a smooth connected variety then the resulting Hilbert Scheme will also be connected.
- Unfortunately, it is also highly singular: Vakil proved Murphy's law for Hilbert Schemes, "There is no singularity so horrible that it cannot occur in a Hilbert scheme".

What is the Hilbert Scheme of points

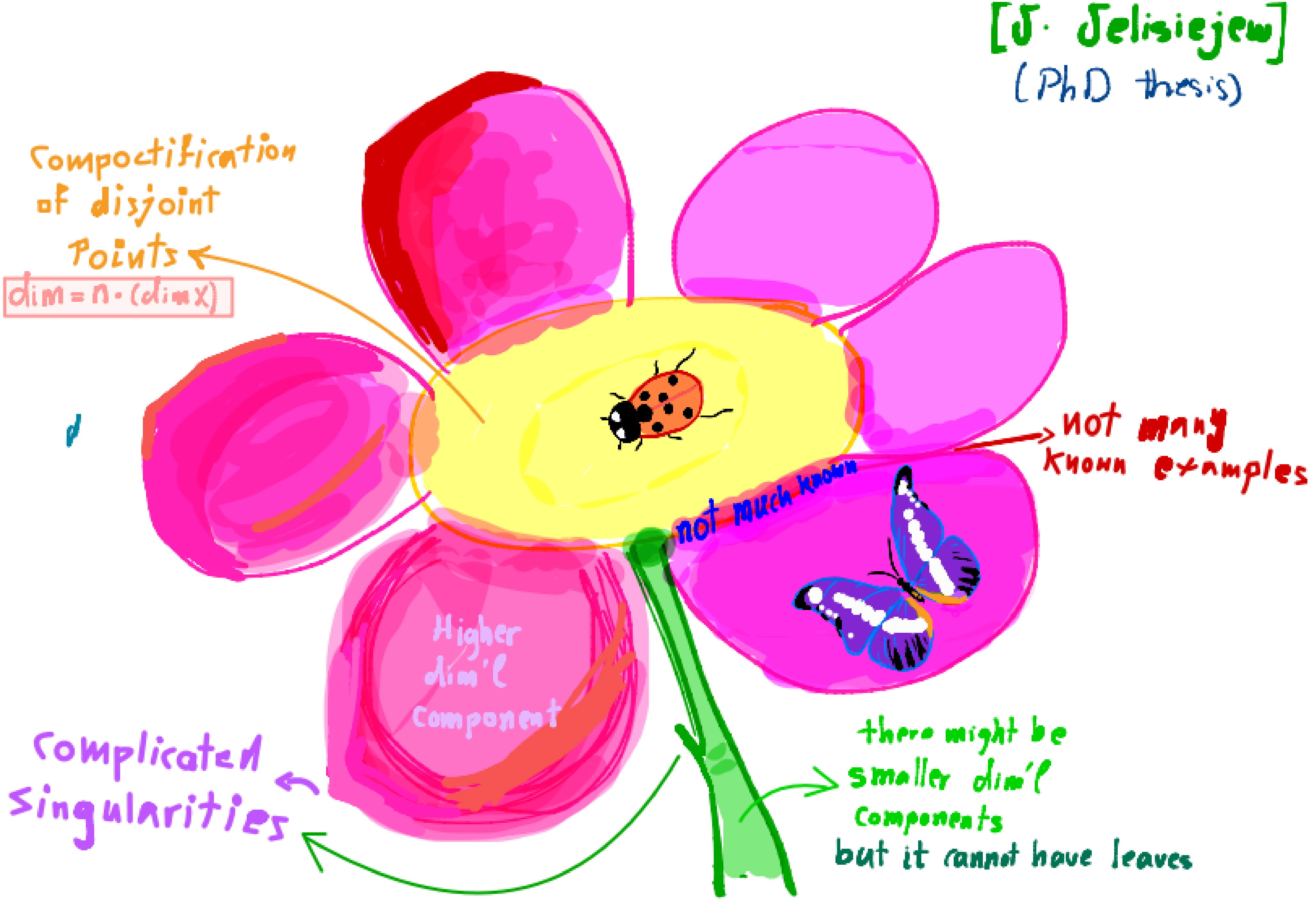
- Consider the set of ways to have l points in N dimensions.
- It is lN dimensional as each point can move in N dimensions.

What is the Hilbert Scheme of points

- We can represent each point in this space by an ideal I of $\mathrm{R} = \mathbb{C}[x_1, x_2, \dots, x_N]$, the set of polynomials which are 0 at all the points.
- We can then represent the properties of original point algebraically.
- The number of points is the dimension of the space R/I.
- The tangent space at an ideal I is given by $Hom_R\left(I, \frac{R}{I}\right)$.

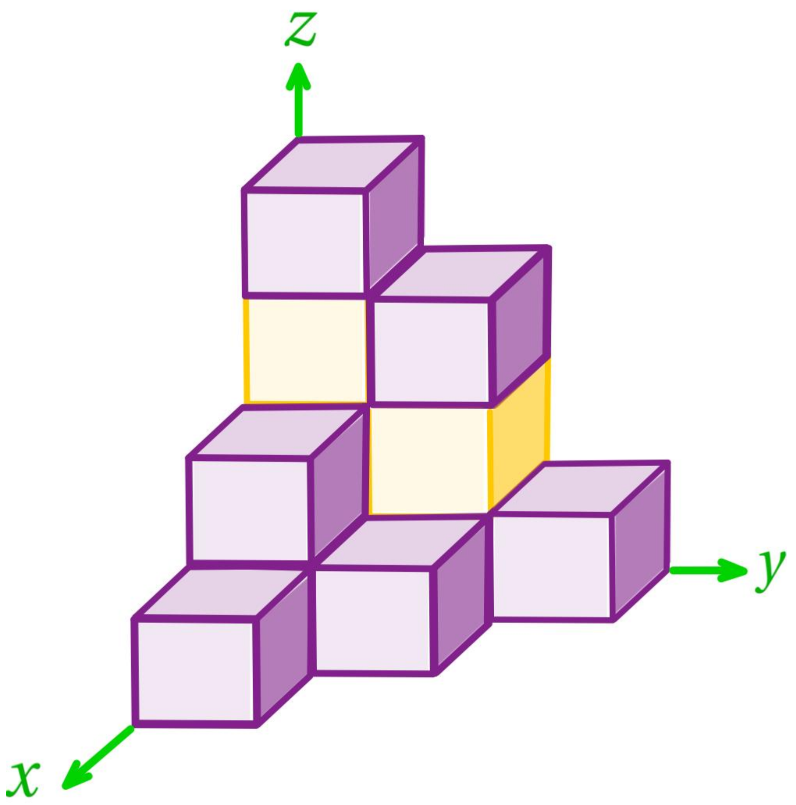
The Hilbert Scheme of points, $Hilb^l \mathbb{A}^N$

- The Hilbert Scheme of points is defined by taking all ideals in R such that R/I is l dimensional.
- But this means that the tangent space at some points has a dimension higher than Nl.


[J. Jelisiejew]
(PhD thesis)
Compactification of disjoint Points
dim = n·(dim X)
not many known examples
not much known
Higher dim'l component
complicated singularities
there might be smaller dim'l components
but it cannot have leaves

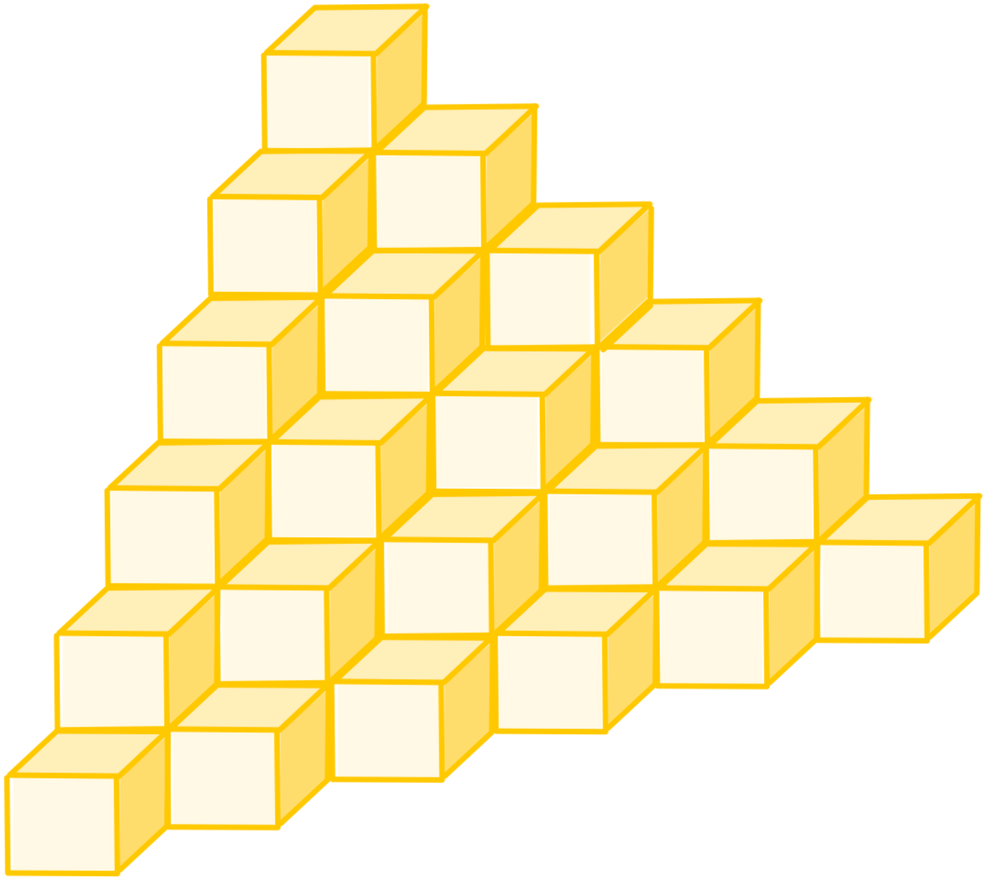
The project

- The project was to investigate how large the tangent space can be for any fixed l specifically focussing on the case $N = 3$.
- The known results for $Hilb^l \mathbb{A}^N$ are that:
  - When $N = 2$ the tangent space always has dimension $2l$.
  - The maximal tangent space is achieved by a monomial ideal.
  - The maximal tangent space is achieved by a Borel-fixed or Strongly Stable ideal.



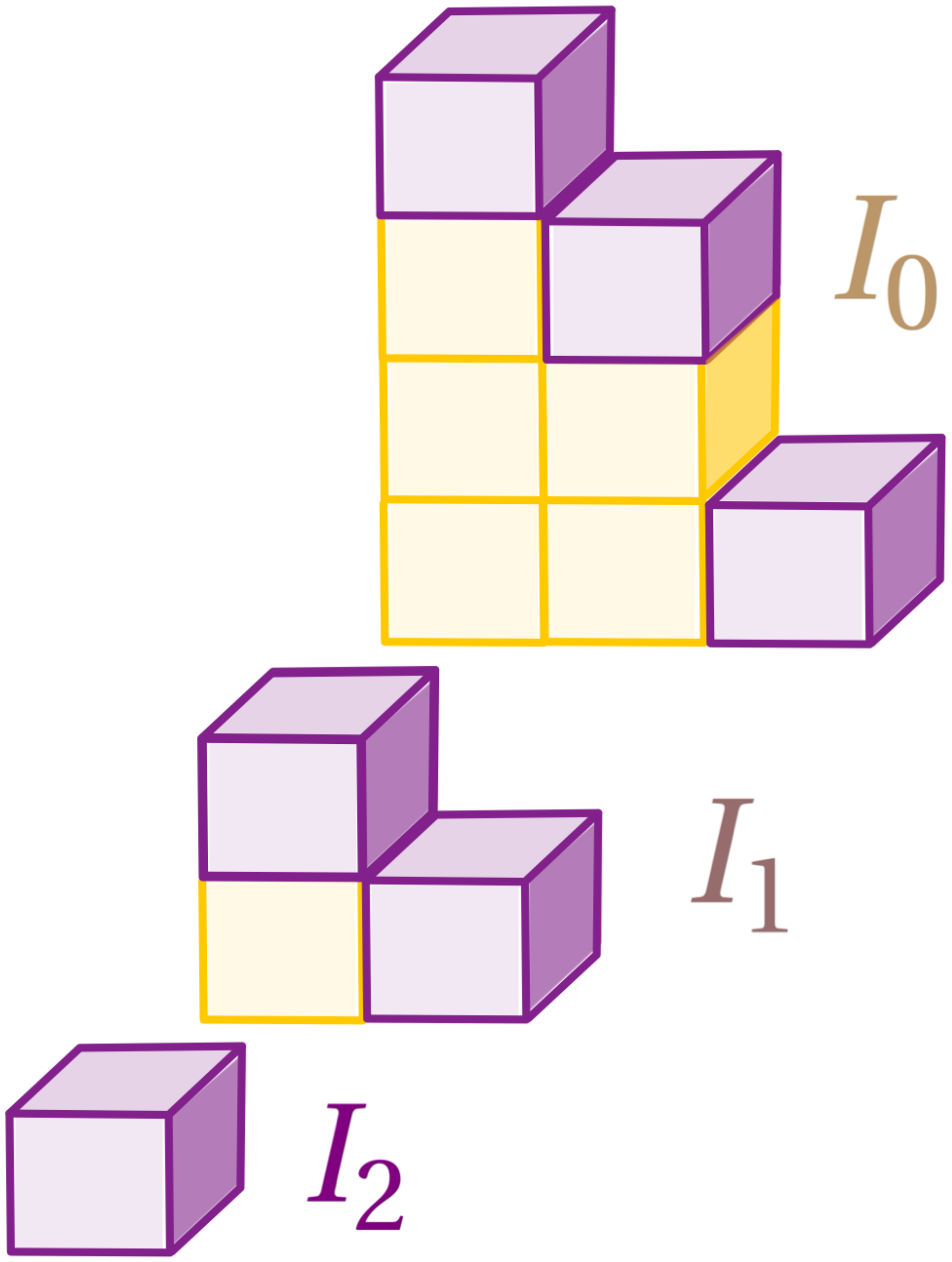
Conjectures

- Briançon and Iarrobino made a conjecture for the maximum tangent space when l is a (hyper)-tetrahedral number.



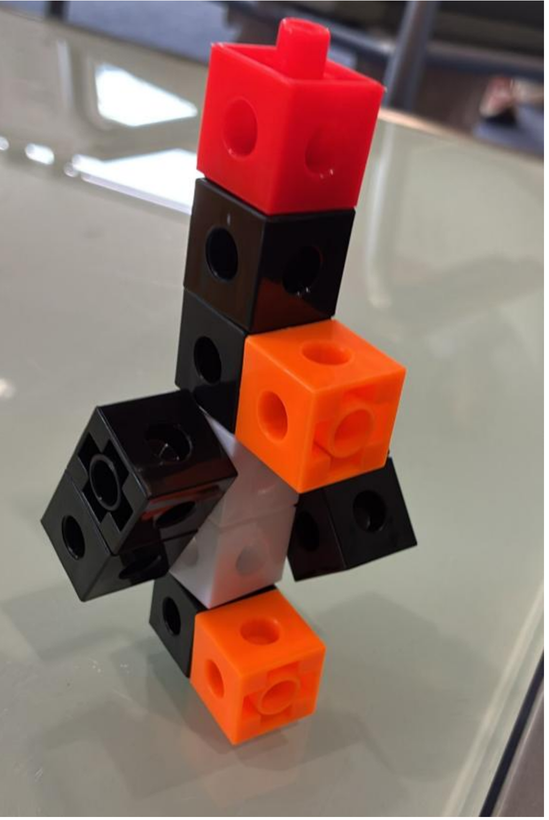
Methods

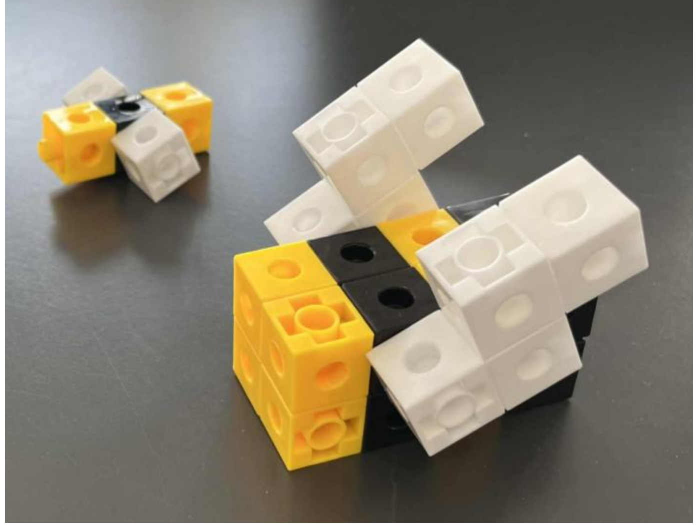
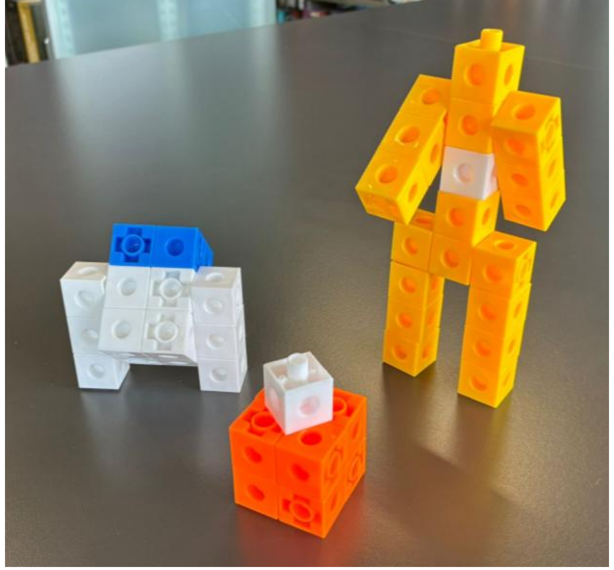
- We considered slicing the ideal and recovered the upper bound achieved by Ramkumar and Sammartano.
- With some extra work, it was possible to refine the upper bound sufficiently to prove the Briançon Iarrobino conjecture in 3 dimensions.



Next Steps

- At the end of the project, we had another idea involving cutting horizontally.
- This could be used for proving the Briançon Iarrobino conjecture in any number of dimensions.



Thank you for listening

Any questions?