Saturation for the Poset $\mathcal{N}$

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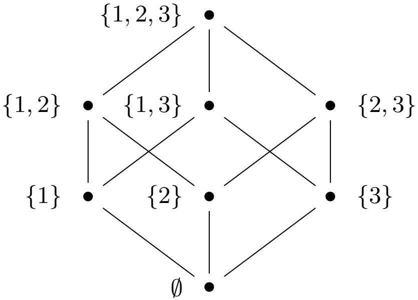
Posets (Partially Ordered Sets)

Partial order: reflexive, antisymmetric, transitive

e.g. power-set of $[n] = \{1, 2, \ldots, n\}$, ordered by set inclusion, represented by Hasse diagram:



Saturation: Graphs

Fix a graph H. A graph G is H-**saturated** iff

- G contains no copy of H
- adding any extra edge to G creates a copy of H

The **saturation number** is

$$\mathrm{sat}(n, H) = \underline{\text{minimum}} \text{ no. of edges in } H\text{-saturated } n\text{-vertex graph}$$

Saturation: Posets

Fix a poset $\mathcal{P}$. Let $\mathcal{F}$ be a family of subsets of $[n]$.

$\mathcal{F}$ is **induced $\mathcal{P}$-saturated** iff

- $\mathcal{F}$ contains no induced copy of $\mathcal{P}$
- adding any other subset to $\mathcal{F}$ creates an induced copy of $\mathcal{P}$

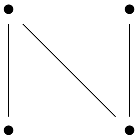
The **induced saturation number** is

$$\text{sat}^*(n, \mathcal{P}) = \underline{\text{minimum}} \text{ size of } \mathcal{P}\text{-saturated family of subsets of } [n]$$

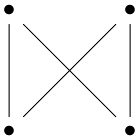
Induced / Non-induced

Induced: preserve all the comparabilities AND incomparabilities.
Non-induced: preserve all the comparabilities.

e.g.



the poset $\mathcal{N}$



the butterfly $\mathcal{B}$

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Induced: preserve all the comparabilities AND incomparabilities.
Non-induced: preserve all the comparabilities.

e.g.

the poset $\mathcal{N}$ the butterfly $\mathcal{B}$

Non-induced copy of $\mathcal{N}$ in blue; but no induced copy in $\mathcal{B}$.

Dichotomy on $\mathrm{sat}^*(n, \mathcal{P})$

Keszegh, Lemons, Martin, Pálvölgyi and Patkós, (2021):

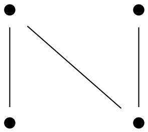
- For any poset $\mathcal{P}$, either $\mathrm{sat}^*(n, \mathcal{P})$ is bounded by a constant, or $\mathrm{sat}^*(n, \mathcal{P}) \geq \log_2 n$ for all n.

They also conjectured:

- For any poset $\mathcal{P}$, either $\mathrm{sat}^*(n, \mathcal{P})$ is bounded by a constant, or $\mathrm{sat}^*(n, \mathcal{P}) \geq n + 1$ for all n.

Goal of Project

To find a linear lower bound on $\mathrm{sat}^*(n, \mathcal{N})$.



The poset $\mathcal{N}$

Components

Let $\mathcal{F}$ be an $\mathcal{N}$-saturated family of subsets of $[n]$.
$\emptyset, [n] \in \mathcal{F}$ always.

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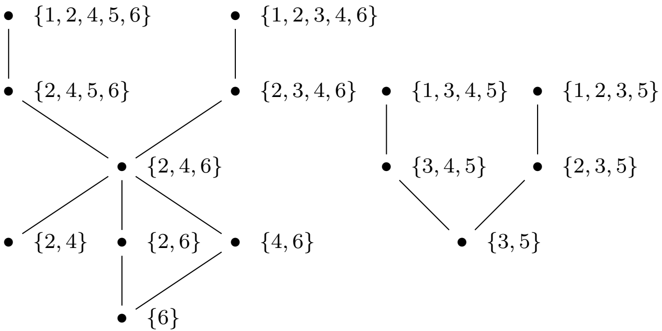

An example of $\mathcal{N}$-saturated family with $n = 6$, excluding $\emptyset, [n]$

Components

Let $\mathcal{G}$ be a component of $\mathcal{F}$, $A \in \mathcal{G}$ maximal, $B \in \mathcal{G}$ minimal. Then $B \subseteq A$.

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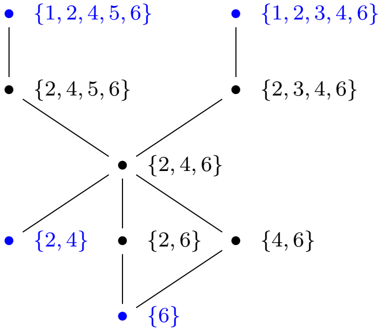

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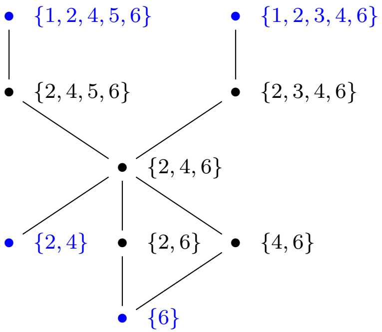
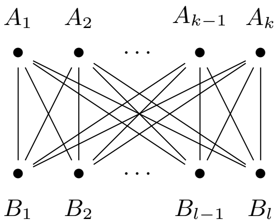


Proof. *Look at the shortest path from* A *to* B.

Midpoints



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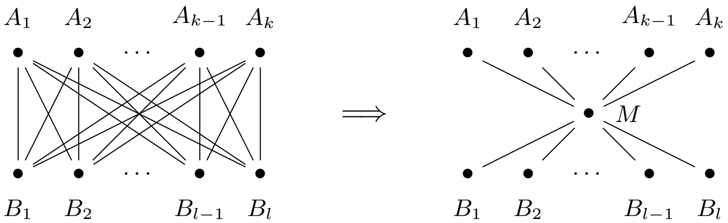

$\mathcal{G}$ has some midpoint between its maximals and minimals (not necessarily unique).

Midpoints

Pick the smallest midpoint M in $\mathcal{G}$.

- Every $X \in \mathcal{G}$ is comparable to M.
- When bounding $|\mathcal{F}|$, can assume $|M| \leq n/2$.

A Graph Theory Argument

Consider a family $\mathcal{F}$ of subsets of $[n]$.
Suppose for all $i \in [n]$, there are $S_i, S_i \cup \{i\} \in \mathcal{F}$, $i \notin S_i$.
Then $|\mathcal{F}| \geq n + 1$.

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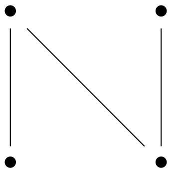
Proof. Construct a graph G with vertex set $\mathcal{F}$, and n edges between each pair $(S_i, S_i \cup \{i\})$, $i \in [n]$.
Can prove G acyclic. So $n = e(G) \leq |\mathcal{F}| - 1$.

Proof Outline

- Recall: some smallest midpoint $|M| \leq n/2$
- $\geq n/2$ singletons $i \notin M$, look at $M \cup \{i\}$
- If $M \cup \{i\} \notin \mathcal{F}$, form an $\mathcal{N}$:



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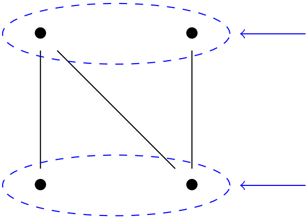
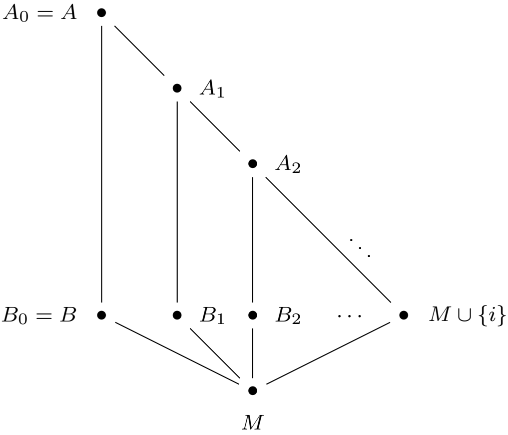


only $\leq |\mathcal{F}| - 2$ singletons have $M \cup \{i\}$ at the top

all the other singletons have $M \cup \{i\}$ at the bottom

Proof Outline



Proof Outline

So $n/2 - |\mathcal{F}| + 2$ singletons satisfy the condition, giving

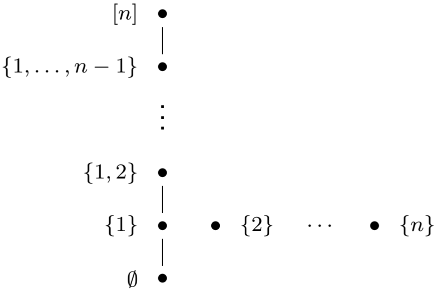
$$|\mathcal{F}| \geq n/4 + 3/2.$$

Therefore

$$\boxed{\text{sat}^*(n, \mathcal{N}) \geq n/4 + 3/2}$$

Remarks

- Improve the current lower bound from $\sqrt{n}$ to $n/4$.
- $\mathrm{sat}^*(n, \mathcal{N}) \leq 2n$ by Ferrara et al. (2017):



So, $\mathrm{sat}^*(n, \mathcal{N})$ grows linearly.