Correlations of Multiplicative Functions

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Acknowledgements

- Thank you to Dr Joni Teräväinen for support!

Preliminary Definitions

Definition

An arithmetic function $f : \mathbb{N} \to \mathbb{C}$ is called multiplicative if

$$f(mn) = f(m)f(n) \quad \text{for all } m, n \text{ coprime.}$$

We say f is completely multiplicative if

$$f(mn) = f(m)f(n) \quad \text{for all } m, n \in \mathbb{N}$$

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- Multiplicative functions are good at dealing with multiplicative information.
- Many problems regarding the distribution of primes can be phrased in the language of multiplicative functions.

- Introduce the Liouville function $\lambda(n) = (-1)^{\Omega(n)}$ where $\Omega(n)$ is equal to the number of prime factors of n counted with multiplicity.

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- Riemann hypothesis is equivalent to the statement

$$\sum_{n \leq x} \lambda(x) = O_{\varepsilon}(x^{\frac{1}{2}+\varepsilon}) \quad \forall \epsilon > 0.$$

Mean values of multiplicative functions

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Conjecture

If $f : \mathbb{N} \to \mathbb{C}$ is multiplicative, then

$$\frac{1}{x}\sum_{n\leq x} f(n) = \prod_{p\leq x}\left(\sum_{k=0}^{\infty} f(p^k)\Big(\frac{1}{p^k} - \frac{1}{p^{k+1}}\Big)\right) + o(1)$$

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- In the case that f is completely multiplicative, we get that

$$\frac{1}{x}\sum_{n\leq x} f(n) \to \prod_{p}\left(1 - \frac{1}{p} + \frac{f(p)}{p}\right)$$

Conjecture

If $f, g : \mathbb{N} \to \mathbb{C}$ are multiplicative, then

$$\frac{1}{x}\sum_{n\le x} f(n)g(n+1) = \prod_{p\le x} M_p(f,g) + o(1)$$

where

$$M_p(f,g) = \sum_{k=0}^{\infty} f(p^k)\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right) + \sum_{k=0}^{\infty} g(p^k)\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right) - 1$$

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- Another interpretation of $M_p(f, g)$:

$$M_p(f,g) = \lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} f_p(n) g_p(n+1)$$

where f_p, g_p are multiplicative functions defined on prime powers by $f_p(q^k) = f(q^k)$ if $q = p$, otherwise $f_p(q^k) = 1$ & similarly for g_p. So correlations of multiplicative functions should satisfy a "local-to-global" property.

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Conjecture (Chowla)

For any distinct naturals $h_1, h_2, ..., h_k$, one has

$$\sum_{n \leq x} \lambda(n + h_1) \cdots \lambda(n + h_k) = o(x).$$

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- $p + 1$ is divisible by 3 with probability $1/\phi(3)$, and divisible by 9 with probability $1/\phi(9)$.

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- So if f is completely multiplicative,

$$\frac{1}{\pi(x)} \sum_{p \le x} f(p+1) \to \prod_{q} \left(1 - \frac{1}{\phi(q)} + \frac{f(q)}{\phi(q)} \right).$$

Conjecture

If $f, g : \mathbb{N} \to \mathbb{C}$ are multiplicative, then

$$\frac{1}{\pi(x)} \sum_{p \le x} f(p+1) g(p+2) = \prod_{q \le x} M_q(f,g) + o(1).$$

where if $q \neq 2$,

$$M_q(f,g) = \sum_{k=0}^{\infty} f(q^k) \Big(\frac{1}{\phi(q^k)} - \frac{1}{\phi(q^{k+1})} \Big) + \sum_{k=0}^{\infty} g(q^k) \Big(\frac{1}{\phi(q^k)} - \frac{1}{\phi(q^{k+1})} \Big) - 1$$

and

$$M_2(f,g) = \sum_{k=0}^{\infty} f(2^k) \Big(\frac{1}{\phi(2^k)} - \frac{1}{\phi(2^{k+1})} \Big)$$

Structure of Multiplicative Functions

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Definition

Let $f, g : \mathbb{N} \to \mathbb{U}$ be multiplicative functions. Define the pretentious distance

$$\mathbb{D}(f, g; y, x) := \Big(\sum_{y<p\leq x} \frac{1 - \mathrm{Re}(f(p)\overline{g(p)})}{p} \Big)^{\frac{1}{2}}.$$

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Set $\mathbb{D}(f, g; x) = \mathbb{D}(f, g; 1, x)$.

- Turns out $\mathbb{D}$ satisfies the triangle inequality.
- This is not quite a metric on the space of multiplicative functions taking values in the unit disc: $\mathbb{D}(f, g; \infty) = 0 \not\Rightarrow f = g$.

- We call a multiplicative function pretentious if $\mathbb{D}(f, n^{it}; \infty) < \infty$ for some $t \in \mathbb{R}$.

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Theorem (Delange)

Let $f : \mathbb{N} \to \mathbb{U}$ be multiplicative. If $\mathbb{D}(f, 1; \infty) < \infty$ then

$$\frac{1}{x}\sum_{n\leq x} f(n) = \prod_{p\leq x}\left(\sum_{k=0}^{\infty} f(p^k)\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right)\right) + o(1)$$

Theorem (Klurman)

Let $f, g : \mathbb{N} \to \mathbb{U}$ be pretentious multiplicative functions, and let $P, Q \in \mathbb{Z}[x]$ be two polynomials. Under certain natural hypotheses,

$$\frac{1}{x} \sum_{n \leq x} f(P(n))g(Q(n))$$

tends to the quantity we expect it to tend to. In the case $\mathbb{D}(f, 1; \infty), \mathbb{D}(g, 1, \infty) < \infty$,

$$\frac{1}{x} \sum_{n \leq x} f(P(n))g(Q(n)) = \prod_{p \leq x} M_p(f, g) + o(1)$$

where

$$M_p(f, g) = \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f_p(P(n))g_p(Q(n)).$$

Theorem

Let $f, g : \mathbb{N} \to \mathbb{U}$ be pretentious multiplicative functions, and let $a, b \in \mathbb{N}$ be distinct. Then

$$\frac{1}{\pi(x)} \sum_{p \leq x} f(p+a)g(p+b)$$

tends to the quantity we expect it to tend to. In the case $\mathbb{D}(f, 1; \infty), \mathbb{D}(g, 1, \infty) < \infty$,

$$\frac{1}{\pi(x)} \sum_{n \leq x} f(p+a)g(p+b) = \prod_{q \leq x} M_q(f, g) + o(1)$$

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Primes in Arithmetic Progressions

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- Key fact used in proofs:

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Theorem (Prime Number Theorem in Arithmetic Progressions)

Let $a, d \in \mathbb{N}$ be such that $(a, d) = 1$. Then

$$\pi(x; d; a) \sim \frac{\mathrm{Li}(x)}{\phi(d)}$$

as $x \to \infty$.

- We want an explicit error term for $\pi(x; d; a) - \frac{\mathrm{Li}(x)}{\phi(d)}$

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Theorem (Siegel–Walfisz)

Let $N \in \mathbb{N}$. Then there exists a constant C_N (depending only on N) such that

$$\pi(x; d; a) = \frac{\mathrm{Li}(x)}{\phi(d)} + O\Big(x \exp\big(-C_N \sqrt{\log x}\big)\Big)$$

uniformly for all $d \leq (\log x)^N$, $(a, d) = 1$.

Theorem

Assuming grand Riemann Hypothesis,

$$\pi(x; d; a) = \frac{\mathrm{Li}(x)}{\phi(d)} + O\left(\sqrt{x}\log x\right)$$

for all $(a, d) = 1$, where the implied constant is absolute.

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- The error term here is slightly bigger than $\sqrt{x}$, and is much better than our error term in Siegel-Walfisz which is slightly smaller than x.
- However, grand Riemann hypothesis is still very much out of reach.

Theorem (Bomberi-Vinogradov)

We have

$$\sum_{q \leq Q} \max_{(a,q)=1} \left| \pi(x; q; a) - \frac{x}{\phi(q)} \right| \ll_A x(\log x)^{-A}$$

for any $A \geq 0$, where $Q = \sqrt{x}(\log x)^{-B}$ with $B = B(A)$.

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- Divide both sides by Q to see that on average, our error term is $\sim \sqrt{x}$, so differs from GRH by some factors of log.
- This is a very powerful & often satisfactory substitute for GRH.

Sieve Theory

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Problem

Let $A \subseteq \mathbb{Z}$ be a finite set of integers. Suppose there exists a multiplicative function $g : \mathbb{N} \to [0,1]$ and $R_d \in \mathbb{R}$ such that

$$\#\{n \in A \mid n \equiv 0 \pmod{d}\} = g(d)|A| + R_d$$

for all squarefree d. Denote

$$S(A,z) = \#\{n \in A \mid p \nmid n \text{ for all } p \leq z\}$$

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- Simplest sieve is the sieve of Eratosthenes-Legendre, based on Eratosthenes' idea on how to find primes.
- Selberg sieve is a very powerful sieve which is very effectively at obtaining upper bounds.

References

- https://www.youtube.com/watch?v=t_plilnbAtM
- https://arxiv.org/pdf/1603.08453