Surfaces, Graphs, and Commutators

Mathematics can be tough, even for experienced mathematicians. Some questions remain unsolved for decades, while others require very complex tools to approach. But if you ask me what the simplest kind of math is, I'd say it's playing with the information you already have. This playful approach is often part of a field called combinatorics.

In our research, we tackled a problem in geometric group theory by translating it into a combinatorial puzzle. Specifically, we aimed to find an explicit upper bound for $M(F)$, the maximum number of distinct ways an element g in a non-abelian free group F can be written as a commutator $[g,h]=ghg^{-1}h^{-1}$, up to a relationship called the "fractional Dehn twist."

To provide some context, a free group is a group without relations, and the commutator $[g,h]$ measures how far g and h fail to commute. The function $num(g)$ counts how many different ways g can be expressed as a commutator under this equivalence, and $M(F)$ takes the maximum of these counts across all elements of F.

Lyndon and Wicks showed in 1981 that $M(F)\geq 2$, which is the best-known lower bound. The first upper bound comes from a general theorem of Sela from around 2000, which implies that $M(F)$ is finite. However, the proof is non-constructive and gives no finite upper bound. Our goal was to fill this gap by finding a concrete upper bound.

We followed an unpublished method by Louder, which uses topology to transform the problem into one about "pieces" on a Möbius band. A Möbius band is a surface formed by giving a strip of paper a half-twist and connecting the ends. This allowed us to rethink the problem in terms of these pieces, which represent possible commutators. You can see pieces as horizontal lines that have a special position on this square. The pieces are constrained in how they interact—they cannot move freely or grow indefinitely.

Each solution to the original problem corresponded to different configurations of these pieces on a square. Interestingly, if there are enough pieces, they start forming a diamond-shaped "carpet" across the surface, representing a key pattern. This structure limits the number of pieces that can appear simultaneously, which was central to our proof of an upper bound for $M(F)$.

By carefully analyzing these interactions and the diamond-like carpet formations, we showed that the number of these pieces is limited, leading us to establish an explicit upper bound for $M(F)$.

What I love about this project is that we turned a complex problem into something playful and manageable. We didn't rely heavily on advanced mathematical techniques but instead used creativity, persistence, and a combinatorial approach to simplify the problem and make real progress.