Constructing algebraic theories via distributive laws
Research Summary

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When we study algebra, we normally study a single type of algebraic structure at a time. For example, we might study groups, or real vector spaces, or rings. But what if we could study how these different types of algebra are related and interconnected?

Enter: *monads*. Monads give us a way to encapsulate the key essence of these *algebraic theories* – e.g. the theory of groups, or the theory of rings – into tidy mathematical parcels. I'm going to write $\mathbf{Sets}$ for the *category of all sets*. You don't have to know precisely what this means – you can just think of it as the collection of all sets! For our purposes, a *monad* will be a *functor* – similarly, you can think of this as a function – of the form $T : \mathbf{Sets} \to \mathbf{Sets}$, along with some extra structure. That is, T takes in a set X, and gives us back another set $T(X)$.

Let's look at some examples. We have the *free abelian group monad* $A : \mathbf{Sets} \to \mathbf{Sets}$, which takes in a set X, and gives us the underlying set of the free abelian group generated by X. E.g. $\{a, b, c\}$ gets sent to

$$\{0,\ a,\ b,\ c,\ 2a,\ a - 7b,\ -a + b + 4c,\ \dots\}.$$

Here's another example: the *free monoid monad*. A monoid is an algebraic structure like a group, but it doesn't care about inverses. Similarly to the above, this gives us a monad $M : \mathbf{Sets} \to \mathbf{Sets}$, which takes in a set X, and gives us the underlying set of the free monoid generated by X. E.g. $\{a, b, c\}$ gets sent to

$$\{1,\ a,\ b,\ c,\ aa,\ abba,\ bbbac,\ \dots\}.$$

There's another property of monads that is important. Because they map from $\mathbf{Sets}$ to $\mathbf{Sets}$, we can iterate them! E.g. we can form $MM(X)$. We then have a way to get a function $MM(X) \to M(X)$ between sets, by 'simplifying'. That is: an element of $MM(X)$ is just a string of elements of $M(X)$, which is just a string of elements of X. E.g. $(abbc)(1)(c) \in MM(\{a, b, c\})$, and we can turn this into an element of $M(\{a, b, c\})$ by just removing the brackets: $abbac$.

An important feature of monads is that every monad T comes with these functions $TT(X) \to T(X)$, one for each set X. Note that whilst T is a functor (it takes in sets and gives out sets), this function $TT(X) \to T(X)$ is a familiar function between sets: it takes in elements of the set $TT(X)$, and gives out elements of the set $T(X)$.

We might also ask if we can compose monads. That is, can $AM : \mathbf{Sets} \to \mathbf{Sets}$ be thought of as a monad? Well, if it can, then we need a way to simplify $AMAM(X) \to AM(X)$. It would be nice if we could do something like transforming $\lambda : MA \to AM$. We could then do

$$AMAM(X) \xrightarrow{\lambda} AAMM(X) \xrightarrow{\text{simplify}} AM(X)$$

where the simplify arrow represents simplifying for A and M individually. Such a λ is called a *distributive law*.

Let's look at an example:

$$\begin{aligned} \lambda : \qquad & MA \to AM \\ & (a+b)(c+d) \mapsto ac + ad + bc + bd \end{aligned}$$

This is the familiar distributivity of multiplication over addition! Whilst this is familiar, it is a highly nontrivial fact that this is the *only* distributive law $MA \to AM$: this was proved by Zwart and Marsden in 2018. My research has involved investigating features of distributive laws that restrict the possible forms they can take.

There's so much exciting mathematics going on behind the scenes here. If you're interested in learning more, try a web search for 'universal algebra', 'monads', or 'distributive laws for monads'. Monads and distributive laws are part of a very abstract area of mathematics called *category theory*, which is well worth your time to look into! Learning a little category theory can give you many different insights into the mathematics you have seen before.

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