Philippa Fawcett Internship Programme 2024

Asymptotic Decomoposition of a Scalar Field in de Sitter Space

Eliza Somerville | Supervised by Dr Grigalius Taujanskas

Our modern understanding of the universe is derived from Einstein's theory of general relativity, which describes gravity as a geometric property of spacetime. The theory is underpinned Einstein's field equations,

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi T_{ab},$$

which relate the curvature of spacetime on the left-hand side, to the stress-energy-momentum of spacetime on the right-hand side. In contrast to other physical theories, general relativity does not have a fixed background on which a well-defined dynamical variable evolves, so we are required to construct the evolution of the metric and of the spacetime on which the metric is defined at the same time. This makes Einstein's equations extremely difficult to solve.

This severe difficulty means that in order to make progress in our understanding of the theory, it is often necessary to choose a fixed background spacetime and study the evolution of other dynamical variables on this background. Some particularly simple examples of spacetimes that are solutions of Einstein's equations are the *maximally symmetric spacetimes*, of which there are three: *Minkowski space* has a vanishing cosmological constant ($\Lambda = 0$), *de Sitter space* has a positive cosmological constant ($\Lambda > 0$), and *anti-de Sitter space* has a negative cosmological constant ($\Lambda < 0$). Of course, we are particularly interested in spacetimes that share similarities with the real universe we inhabit. As recent astronomical observations have indicated that the cosmological constant of our universe is in fact positive, we therefore have a particular interest in studying de Sitter space.

One area of great interest in general relativity is the study of *metric scattering*, which is the effect of curved spacetime on the asymptotic behaviour of fields. Much of the work in this field has been enabled by the use of the *conformal method*. This method is based on Roger Penrose's discovery in the 1960s of a way of compactfying certain spacetimes by performing a conformal transformation of the metric and attaching a boundary $\mathscr{I}$ called *null infinity*. This enables us to study the behaviour of the field on $\mathscr{I}$ using local differential geometry in the conformally compactified spacetime, and then translate this into conditions on the asymptotic behaviour of the field on the physical spacetime.

This project consisted of a study of the asymptotic behaviour of a scalar field that obeys the linear conformal wave equation on four-dimensional de Sitter space from several different angles. The conformal method was used to investigate the existence of an asymptotic decomposition of the scalar field into exponentially-decaying components. Following the approach which was used to analyse the nonlinear Maxwell–scalar field system in [1], the first three coefficients in the asymptotic expansion for the simpler linear system were derived. From the pattern observed, a conjecture was made for the general form of an arbitrary coefficient in the expansion of the scalar field in the linear case. Further work is needed to derive a similar expansion for the full nonlinear Maxwell-scalar field system.

The same problem was also studied using a different approach, by developing the method from [2] involving *quasinormal modes* – objects that describe the behaviour of fields that decay in time a via series expansion of resonant terms. This method has some similarities to the conformal method, as it again uses a compactification to enable the use of tools such as Taylor expansions at the cosmological horizon. However, it is a more direct approach, and one that gives a full asymptotic expansion in a way that allows us to easily read of the quasinormal modes and the corresponding mode solutions. It was ultimately found that the expression obtained via this method agreed with the conjectured formula for the coefficients of the asymptotic expansion of the scalar field obtained via the conformal method, which provided an important validation of these results for the case of the linear wave equation.

References

[1] G. Taujanskas. 'Conformal scattering of the Maxwell-scalar field system on de Sitter space'. *Journal of Hyperbolic Differential Equations* 16.04 (2019), pp. 743–791. arXiv: 1809.01559 [math.AP].

[2] P. Hintz and Y. Xie. 'Quasinormal modes and dual resonant states on de Sitter space'. *Physical Review D* 104.6 (2021). arXiv: 2104.11810 [gr-qc].