Asymptotic Decomposition of a Scalar Field in de Sitter Space

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in collaboration with
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Einstein's equations of general relativity:

$$\underbrace{R_{ab} - \frac{1}{2} R g_{ab} + \overset{\text{Cosmological constant}}{\lambda} \overset{\text{Spacetime metric}}{g_{ab}}}_{\text{Curvature of spacetime}} = \underbrace{8\pi G\, T_{ab}}_{\text{Energy-momentum of spacetime}}.$$

De Sitter space = Maximally symmetric solution of Einstein's equations with positive cosmological constant.

Motivation

Einstein's equations of general relativity:

$$\underbrace{R_{ab} - \frac{1}{2} R g_{ab} + \overset{\text{Cosmological constant}}{\lambda} \overset{\text{Spacetime metric}}{g_{ab}}}_{\substack{\text{Curvature of} \\ \text{spacetime}}} = \underbrace{8\pi G\, T_{ab}}_{\substack{\text{Energy-momentum} \\ \text{of spacetime}}}.$$

De Sitter space = Maximally symmetric solution of Einstein's equations with positive cosmological constant.

Goal: To investigate the existence of a conjectured asymptotic expansion for the charged scalar field on de Sitter space:

$$\phi \sim \varphi_1 \, e^{-Ht} + \varphi_2 \, e^{-2Ht} + \varphi_3 \, e^{-3Ht} + \dots$$

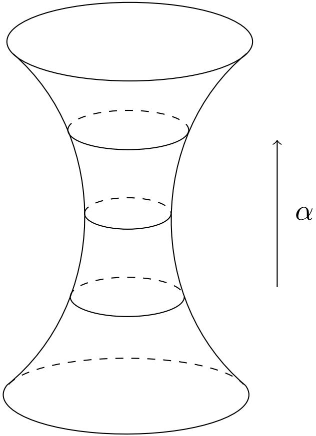
De Sitter Space

De Sitter space dS_4 may be defined as the hyperboloid

$$|x|^2 - x_0^2 = \frac{1}{H^2}$$

in $(4+1)$-dimensional Minkowski space

$$\eta_5 = \mathrm{d}x_0^2 - \mathrm{d}|x|^2 - |x|^2 g_{\mathbb{S}^3}.$$



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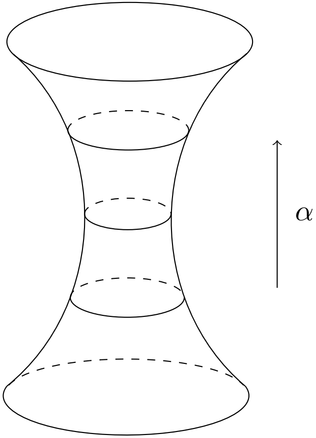
Defining

$$x_0 = \frac{1}{H}\sinh(H\alpha), \qquad |x| = \frac{1}{H}\cosh(H\alpha),$$

the metric η_5 descends to the metric g on dS_4,

$$g = \mathrm{d}\alpha^2 - \frac{1}{H^2}\cosh^2(H\alpha) g_{\mathbb{S}^3}.$$



Conformal Compactification

To study the asymptotic structure of a spacetime $(\mathscr{M}, g)$ at infinity, we make the *conformal transformation*

$$g_{ab} \to \hat{g}_{ab} = \Omega^2 g_{ab}$$

Conformal factor, $\to 0$ asymptotically

This brings infinity to a finite region.

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Attach to $\mathscr{M}$ a boundary $\mathscr{I} := \{\Omega = 0\}$ and get a new spacetime

$$\hat{\mathscr{M}} = \mathscr{M} \cup \mathscr{I}$$

Asymptotic considerations in physical spacetime $\mathscr{M}$

$\updownarrow$

Local differential geometry near $\mathscr{I}$ in the rescaled spacetime $\hat{\mathscr{M}}$.

Conformal Compactification of de Sitter Space

$$g = \mathrm{d}\alpha^2 - \frac{1}{H^2}\cosh^2(H\alpha) g_{\mathbb{S}^3}$$

Make the coordinate transformation

$$\tan\left(\frac{\tau}{2}\right) = \tanh\left(\frac{H\alpha}{2}\right)$$

so that the metric becomes

$$g = \frac{1}{H^2\cos^2\tau}\left(\mathrm{d}\tau^2 - g_{\mathbb{S}^3}\right)$$

where $\tau \in (-\pi/2, \pi/2)$.

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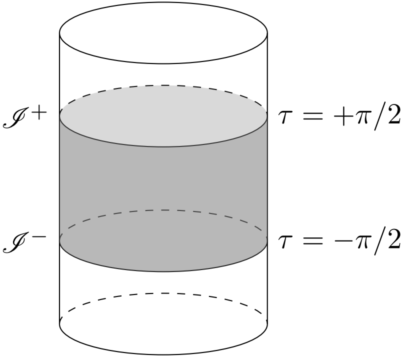
$\uparrow$
Metric on the Einstein cylinder
$\mathbb{R}\times\mathbb{S}^3$

where $\tau \in (-\pi/2, \pi/2)$.



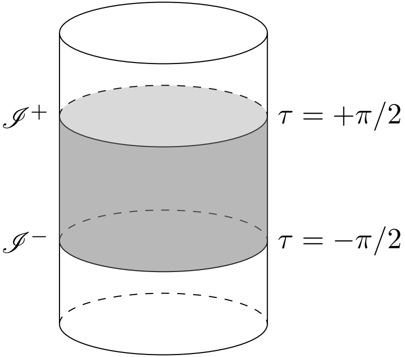
Conformal Compactification of de Sitter Space

$$g = \Omega^{-2}(\mathrm{d}\tau^2 - g_{\mathbb{S}^3}), \quad \Omega = H\cos\tau$$

We can attach to $(-\pi/2, \pi/2) \times \mathbb{S}^3$ the boundary

$$\mathscr{I} := \{\Omega = 0\} = \{\tau = \pm\pi/2\}$$

and identify compactified de Sitter space $\widehat{\mathrm{dS}}_4$ with $[-\pi/2, \pi/2] \times \mathbb{S}^3$.



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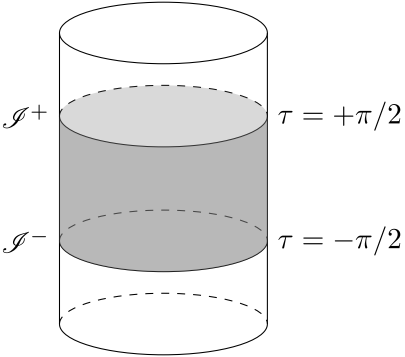
The boundary is the union of the spacelike hypersurfaces

$$\mathscr{I}^+ = \left\{\tau = +\frac{\pi}{2}\right\}, \qquad \mathscr{I}^- = \left\{\tau = -\frac{\pi}{2}\right\}.$$

Future null infinity

Past null infinity



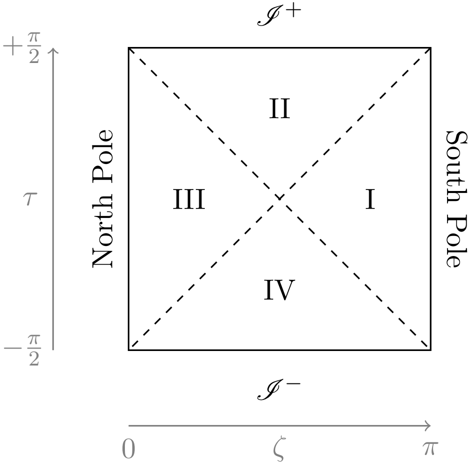
Penrose Diagram for de Sitter Space

$$g = \Omega^{-2}(\mathrm{d}\tau^2 - g_{\mathbb{S}^3}), \quad \Omega = H\cos\tau$$

If we write the three-sphere metric as

$$g_{\mathbb{S}^3} = \mathrm{d}\zeta^2 + (\sin^2\zeta) g_{\mathbb{S}^2}$$

for $\zeta \in [0, \pi]$ and quotient out the SO(3) symmetry group of $g_{\mathbb{S}^2}$, we obtain the Penrose diagram for dS_4.



Static coordinates on dS_4 may be constructed by defining

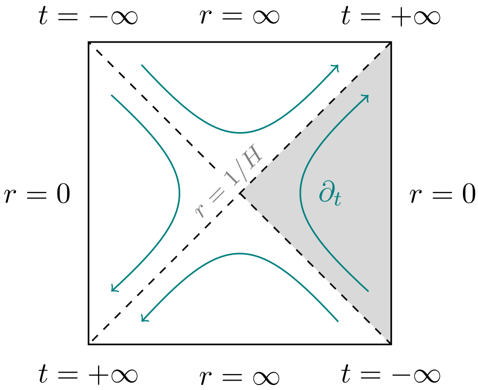
$$r = \frac{\sin\zeta}{H\cos\tau}, \qquad \tanh(Ht) = \frac{\sin\tau}{\cos\zeta}$$

for $\tau \in (-\pi/2, \pi/2)$ and $\zeta \in (0, \pi)$.

Then

$$g = F(r)\mathrm{d}t^2 - F(r)^{-1}\mathrm{d}r^2 - r^2 g_{\mathbb{S}^2},$$

where $F(r) = 1 - H^2 r^2$.



The Conformal Wave Equation

For a generic spacetime $(\mathscr{M}, g)$, the conformal wave equation is

Scalar field

$$\Box\phi + \frac{1}{6}R\phi = 0.$$

$\nabla_a\nabla^a = g^{ab}\nabla_a\nabla_b$

Scalar curvature of spacetime

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Scalar field

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$\nabla_a\nabla^a = g^{ab}\nabla_a\nabla_b$ Scalar curvature of spacetime

Consider the conformal transformation $\hat{g}_{ab} = \Omega^2 g_{ab}$, and choose

$$\hat{\phi} := \Omega^{-1}\phi.$$

Then the wave equation is *conformally invariant*:

$$\Box\phi + \frac{1}{6}R\phi = 0 \quad \Longleftrightarrow \quad \hat{\Box}\hat{\phi} + \frac{1}{6}\hat{R}\hat{\phi} = 0.$$

The Conformal Wave Equation on de Sitter Space

For de Sitter space we have $R = 12H^2$, so that the wave equation on dS_4 is

$$\Box\phi + 2H^2\phi = 0.$$

Under the rescaling

$$\hat{g}_{ab} = \Omega^2 g_{ab}, \qquad \hat{\phi} = \Omega^{-1}\phi, \qquad \text{with} \;\; \Omega = H\cos\tau,$$

this becomes the conformal wave equation on the Einstein cylinder,

$$\hat{\Box}\hat{\phi} + \hat{\phi} = 0.$$

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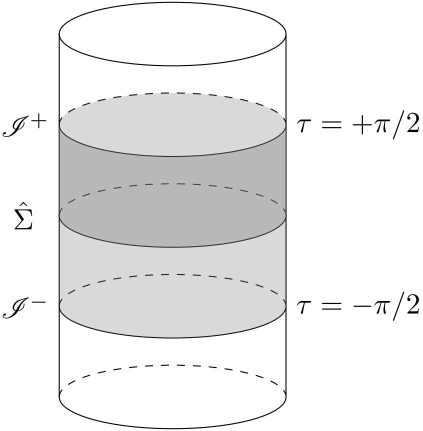
The Conformal Method

$$\text{Estimates for } \hat{\phi} \text{ on compactified spacetime } \widehat{\mathrm{dS}}_4$$
$$\downarrow$$
$$\text{Estimates for } \phi \text{ on physical spacetime } \mathrm{dS}_4$$

Decay Estimate

Estimates for $\hat{\phi}$ on Einstein cylinder
$\rightarrow$ Estimates for ϕ on physical spacetime dS_4.

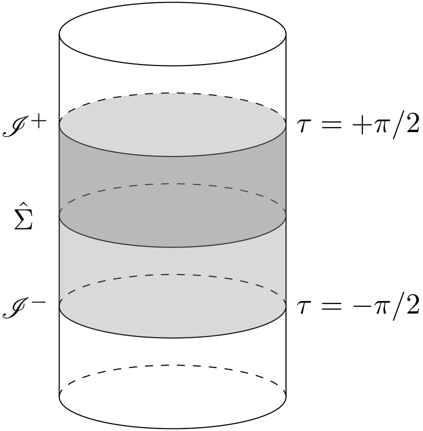


Decay Estimate

Estimates for $\hat{\phi}$ on Einstein cylinder
$\rightarrow$ Estimates for ϕ on physical spacetime dS_4.

For sufficiently regular initial data $(\hat{\phi}, \partial_\tau \hat{\phi})|_{\hat{\Sigma}}$, one can show that

$$|\hat{\phi}| \leq C \quad \text{as} \quad \tau \to \pi/2.$$



Decay Estimate

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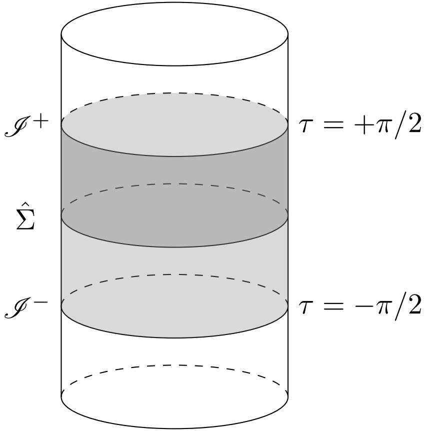
For sufficiently regular initial data $(\hat{\phi}, \partial_\tau \hat{\phi})|_{\hat{\Sigma}}$, one can show that

$$|\hat{\phi}| \leq C \quad \text{as} \quad \tau \to \pi/2.$$

Then since $\phi = \Omega\, \hat{\phi}$,

$$|\phi| \lesssim \Omega \quad \text{as} \quad t \to +\infty.$$

Inequality up to a constant



Decay Estimate in Static Coordinates

In the static coordinates,

$$\Omega = \frac{H}{\cosh(Ht)} \frac{1}{\sqrt{1 - H^2 r^2 \tanh^2(Ht)}} \sim \frac{H\mathrm{e}^{-Ht}}{\sqrt{1 - H^2 r^2}} \quad \text{as} \quad t \to +\infty,$$

so that keeping r fixed, we have

$$|\phi| \lesssim \Omega \lesssim_r \mathrm{e}^{-Ht} \quad \text{as} \quad t \to +\infty.$$

Asymptotic Decomposition of a Scalar Field

We now know that

$$\phi \sim \varphi_1 \mathrm{e}^{-Ht} + \mathcal{O}(\mathrm{e}^{-2Ht}) \quad \text{as} \;\; t \to +\infty.$$

How can we find the coefficient φ_1?

We now know that

$$\phi \sim \varphi_1 \mathrm{e}^{-Ht} + \mathcal{O}(\mathrm{e}^{-2Ht}) \quad \text{as} \;\; t \to +\infty.$$

How can we find the coefficient φ_1?

Relate derivatives on $\widehat{\mathrm{dS}}_4$ to derivatives on dS_4:

$$\Omega \partial_\zeta \hat{\phi} = \frac{\partial t}{\partial \zeta}\partial_t \phi + \frac{\partial r}{\partial \zeta}\partial_r \phi$$
$$= rF(r)^{-1/2}\sinh(Ht)\partial_t \phi + H^{-1}F(r)^{1/2}\cosh(Ht)\partial_r \phi$$

$$\Omega \partial_\tau \hat{\phi} = \frac{\partial t}{\partial \tau}\partial_t \phi + \frac{\partial r}{\partial \tau}\partial_r \phi - \Omega^{-1}(\partial_\tau \Omega)\phi$$
$$= H^{-1}F(r)^{-1/2}\cosh(Ht)\partial_t \phi + rF(r)^{1/2}\sinh(Ht)\partial_r \phi + F(r)^{1/2}\sinh(Ht)\phi$$

Reminder:

$$r = \frac{\sin\zeta}{H\cos\tau}$$
$$\tanh(Ht) = \frac{\sin\tau}{\cos\zeta}$$

Asymptotic Decomposition: First Coefficient

$$\Omega\partial_\zeta\hat{\phi} = rF(r)^{-1/2}\sinh(Ht)\partial_t\phi + H^{-1}F(r)^{1/2}\cosh(Ht)\partial_r\phi$$

$$\Omega\partial_\tau\hat{\phi} = H^{-1}F(r)^{-1/2}\cosh(Ht)\partial_t\phi + rF(r)^{1/2}\sinh(Ht)\partial_r\phi + F(r)^{1/2}\sinh(Ht)\phi$$

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For sufficiently regular initial data, $\partial_\zeta\hat{\phi}$ and $\partial_\tau\hat{\phi}$ have continuous limits on $\mathscr{I}^+$, so

$$|\Omega\partial_\zeta\hat{\phi}|, |\Omega\partial_\tau\hat{\phi}| \lesssim \Omega \lesssim \mathrm{e}^{-Ht} \quad \text{as} \;\; t \to +\infty.$$

Asymptotic Decomposition: First Coefficient

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For sufficiently regular initial data, $\partial_\zeta\hat{\phi}$ and $\partial_\tau\hat{\phi}$ have continuous limits on $\mathscr{I}^+$, so

$$|\Omega\partial_\zeta\hat{\phi}|, |\Omega\partial_\tau\hat{\phi}| \lesssim \Omega \lesssim \mathrm{e}^{-Ht} \quad \text{as} \;\; t \to +\infty.$$

Considering the e^{-Ht} component of ϕ,

$$\varphi_1 := \mathrm{e}^{Ht}\phi,$$

and taking the limit as $t \to +\infty$,

$$0 \approx Hr\partial_t\varphi_1 - H^2r\varphi_1 + F\partial_r\varphi_1,$$

$$0 \approx \partial_t\varphi_1 - H\varphi_1 + HrF\partial_r\varphi_1 + HF\varphi_1.$$

Equality at $t = +\infty$

Asymptotic Decomposition: First Coefficient

$$0 \approx Hr\partial_t\varphi_1 - H^2 r\varphi_1 + F\partial_r\varphi_1,$$
$$0 \approx \partial_t\varphi_1 - H\varphi_1 + HrF\partial_r\varphi_1 + HF\varphi_1$$

Solving this algebraically, we find that $\partial_t\varphi_1 \approx 0$, and

$$H^2 r\varphi_1 \approx F(r)\partial_r\varphi_1.$$

Solving this ordinary differential equation in r, we obtain

$$\varphi_1(r) \approx \frac{1}{\sqrt{F(r)}}\varphi_1(0).$$

Asymptotic Decomposition: Second Coefficient

For the second coefficient, compute

$$\Omega\partial_\zeta^2\hat{\phi}, \qquad \Omega\partial_\zeta\partial_\tau\hat{\phi}, \qquad \Omega\partial_\tau^2\hat{\phi},$$

and define

$$\varphi_2 := \mathrm{e}^{2Ht}(\phi - \varphi_1\mathrm{e}^{-Ht}).$$

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and define

$$\varphi_2 := \mathrm{e}^{2Ht}(\phi - \varphi_1\mathrm{e}^{-Ht}).$$

We find that φ_2 is also independent of t, and obtain the ODE

$$F\partial_r^2\varphi_2 - 4H^2r\partial_r\varphi_2 - 2H^2\varphi_2 \approx 0,$$

which has solution

$$\varphi_2(r) \approx \frac{\varphi_2(0) + r\varphi_2'(0)}{F(r)}.$$

Similarly, for the third coefficient, we compute the third derivatives

$$\Omega \partial_\zeta^3 \hat{\phi}, \qquad \Omega \partial_\zeta^2 \partial_\tau \hat{\phi}, \qquad \Omega \partial_\zeta \partial_\tau^2 \hat{\phi} \qquad \Omega \partial_\tau^3 \hat{\phi},$$

and find that

$$\varphi_3(r) \approx \frac{\varphi_3(0) + r\varphi_3'(0) + r^2 \varphi_3''(0)}{F(r)^{3/2}}.$$

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Similarly, for the third coefficient, we compute the third derivatives

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and find that

$$\varphi_3(r) \approx \frac{\varphi_3(0) + r\varphi_3'(0) + r^2\varphi_3''(0)}{F(r)^{3/2}}.$$

We thus have the asymptotic decomposition

$$\begin{aligned}\phi &\sim \varphi_1\,\mathrm{e}^{-Ht} + \varphi_2\,\mathrm{e}^{-2Ht} + \varphi_3\,\mathrm{e}^{-3Ht} + \dots \\ &\sim \frac{\varphi_1(0)}{F(r)^{1/2}}\mathrm{e}^{-Ht} + \frac{\varphi_2(0) + r\varphi_2'(0)}{F(r)}\mathrm{e}^{-2Ht} + \frac{\varphi_3(0) + r\varphi_3'(0) + r^2\varphi_3''(0)}{F(r)^{3/2}}\mathrm{e}^{-3Ht} + \dots\end{aligned}$$

Conclusion

The conformal method can be used to study the asymptotic structures of spacetimes.

We investigated an asymptotic decomposition of a scalar field on de Sitter space,

$$\phi \sim \varphi_1 \, \mathrm{e}^{-Ht} + \varphi_2 \, \mathrm{e}^{-2Ht} + \varphi_3 \, \mathrm{e}^{-3Ht} + \dots$$

and found the coefficients up to $\mathcal{O}(e^{-3Ht})$.

From the observed pattern, we conjecture that

$$\varphi_n(r) \approx \frac{\varphi_n(0) + r\varphi_n'(0) + r^2\varphi_n''(0) + \cdots + r^{n-1}\varphi_n^{(n-1)}(0)}{F(r)^{n/2}}.$$

Conclusion

The coefficients φ_n derived using the conformal method agree with

- Calculations using quasinormal modes on dS_4,
- Direct solution of the PDEs derived from the conformal wave equation.

The asymptotic expansion using the conformal method also holds for the non-linear Maxwell-scalar field system,

$$\nabla^b F_{ab} = \mathrm{Im}(\bar{\phi} D_a \phi),$$

$$D^a D_a \phi + \frac{1}{6} R\phi = 0.$$

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