The Double-Double Ramification Cycle

Intersection theory on the moduli space of curves

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Overview

Algebraic geometry: the study of spaces defined by polynomial eqns.

1. **Intersection theory**.
2. $\overline{\mathcal{M}}_{g,n}$, the **moduli space of curves**.
3. Intersection theory revisited.
4. The **double-double ramification cycle**.

varieties = spaces.

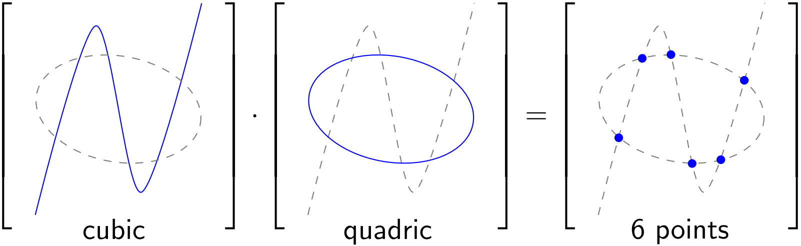
curves = 1-dim spaces
= Riemann surfaces.

Ultimately, compute *intersection numbers*:

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_{g-1} \mathrm{DDR}_g(A, B) \prod_i \psi_i^{\alpha_i} = \dots$$

Intersection Theory

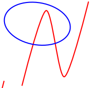
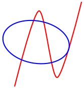
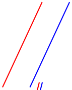
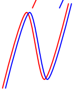
Bézout's Theorem (1779):



Pairing: curves $\times$ curves $\xrightarrow{\text{intersection}} \mathbb{Z}$ is multiplicative.

Intersection Theory

Problems:

1. Less intersections 
2. Tangencies 
3. Parallel asymptotes 
4. Self-intersections 

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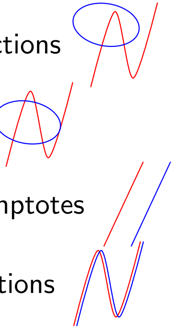

Solutions:

1. $\mathbb{R} \xrightarrow{\text{alg. closure}} \mathbb{C}$
2. Count multiplicities
3. $\mathbb{C}^2 \xrightarrow{\text{compactify}} \mathbb{CP}^2$
4. Allow "deformations":

rational equivalence

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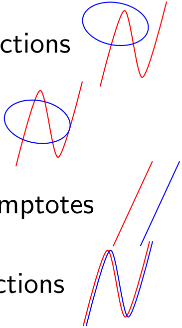

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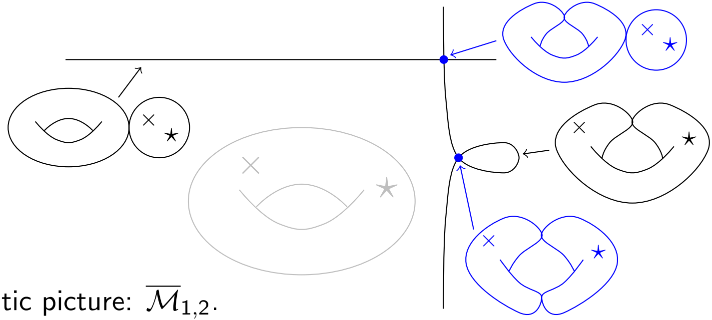
Bézout on $\mathbb{P}^2$: $(\text{curves}/\sim) \times (\text{curves}/\sim) \xrightarrow{\cap\text{-pairing}} \mathbb{Z}$.

Int. theory: **product structure** on (subvarieties/rational equivalence).

The Moduli Space of Curves

$\overline{\mathcal{M}}_{g,n}$ = **moduli (param.) space of smooth/nodal stable curves** of genus g with n marked points.

- Concept: Riemann (1857). $\dim = 3g - 3 + n$.
- Existence: Deligne-Mumford (1969).



Schematic picture: $\overline{\mathcal{M}}_{1,2}$.

genus-0 curve $= \mathbb{P}^1 \sim$ sphere

genus-1 curve $\sim$ torus

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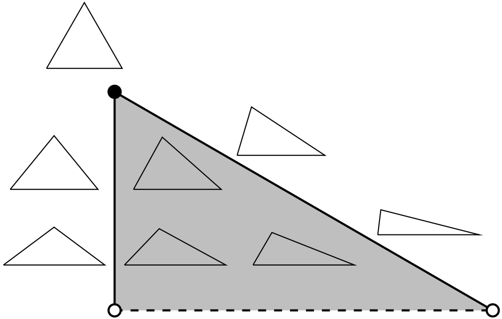
$\{\text{triangles}\} =$ 

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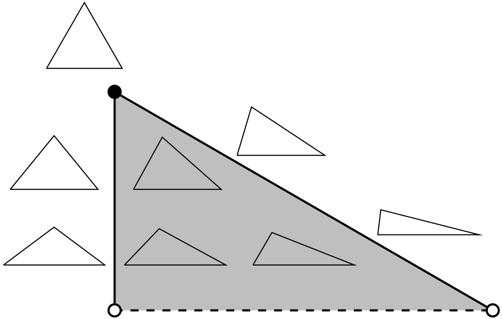
$\{\text{triangles}\} =$ 

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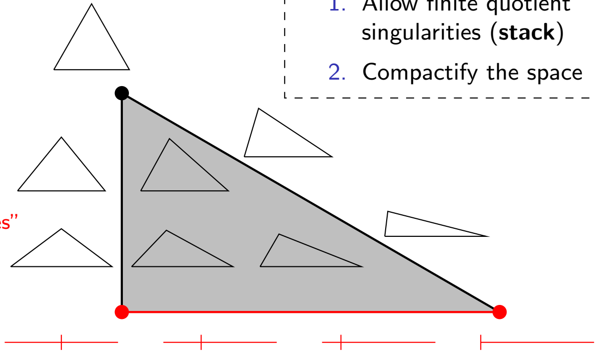
$\{\text{triangles*}\} =$

*including "flat triangles"



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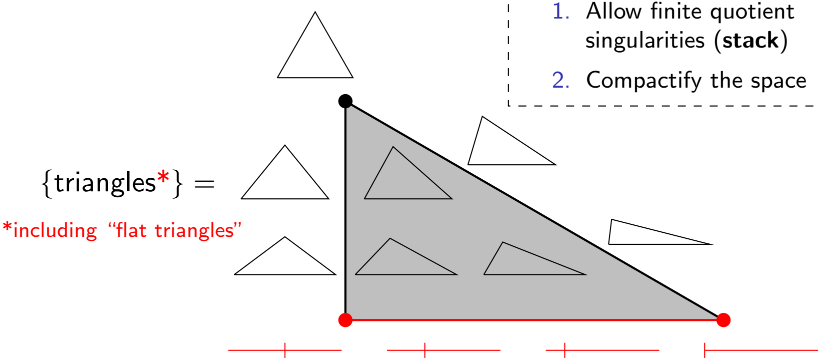
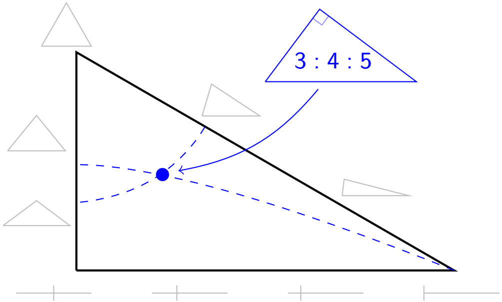


Moduli space of triangles: $\mathcal{T} \approx$ 2-dimensional *algebraic stack*.

Intersection Theory on Moduli Spaces: A Toy Model

Enumerative problem $\longleftrightarrow$ intersection theory on moduli space:

$$\left(\begin{array}{l}\#\text{ triangles that are}\\ \quad 1.\ \text{Right-angled}\\ \quad 2.\ \text{Area} = \frac{(\text{Perimeter})^2}{24}\end{array}\right) =$$



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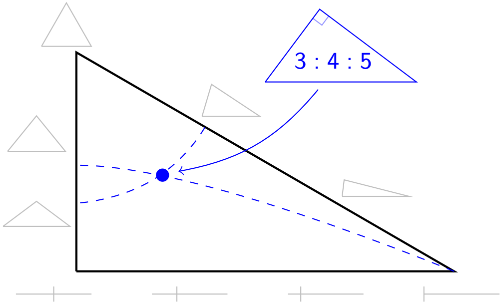


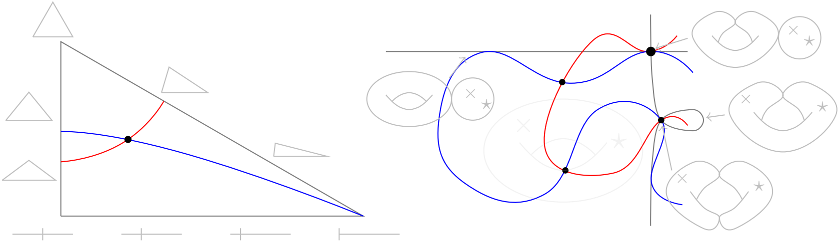
Bézout: $\#(\deg 2 \cap \deg 4) = 8$. Why do we only see 1 solution?

1. C_2-symmetry
2. Nonphysical solns: $[x:y:z] = [1:-3.82:-3.95], [0:1:-1]_{\text{mult. }2}$

Enumerative Geometry: Intersection Theory on $\overline{\mathcal{M}}_{g,n}$

Enumerating triangles $\longleftrightarrow$ intersection theory on $\mathcal{T}$.

Enumerating curves $\longleftrightarrow$ intersection theory on $\overline{\mathcal{M}}_{g,n}$.



E.g. "How many rational cubics on $\mathbb{P}^2$ pass through 8 given points?"

Intersection Numbers: Measuring the Shape of Varieties

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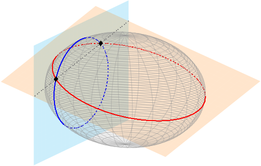
If $\dim Y = k$, $Y \cap (k$ codim-1 subvarieties$)$ = **intersection number**:

$$\int_X [Y] \cdot [Z_1] \cdot \ldots \cdot [Z_k] = \#(Y \cap Z_1 \cap \cdots \cap Z_k)$$

$\alpha \mapsto \int_X \alpha \cdot \prod_i [Z_i]$ is a "test function": (subvarieties$/\sim$) $\to \mathbb{Z}$..

Two Examples: Intersection Numbers on $\mathbb{P}^n$ and $\mathbb{P}^n \times \mathbb{P}^m$

$X = \mathbb{P}^n$, $Y \subset X$ of dimension k:



$$\deg Y = \int_{\mathbb{P}^n} [Y] \cdot [H]^k$$

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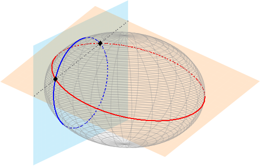

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$X = \mathbb{P}^n \times \mathbb{P}^m$, $Y \subset X$ of dimension k: for each $a = 0, 1, \ldots, k$,

$$\deg_{a,k-a} Y = \int_{\mathbb{P}^n \times \mathbb{P}^m} [Y] \cdot [H_1]^a \cdot [H_2]^{k-a}$$

where $[H_1] = [\text{hyperplane} \times \mathbb{P}^m]$ and $[H_2] = [\mathbb{P}^n \times \text{hyperplane}]$.

E.g. $k = 3$: $(\deg_{0,3} Y, \deg_{1,2} Y, \deg_{2,1} Y, \deg_{3,0} Y)$ determines $[Y]$.

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Each $Y \subset \overline{\mathcal{M}}_{g,n}$ may be "tested against" **psi classes** $\psi_1, \ldots, \psi_n$:

$$\int_{\overline{\mathcal{M}}_{g,n}} [Y] \cdot \psi_1^{e_1} \ldots \psi_n^{e_n}$$

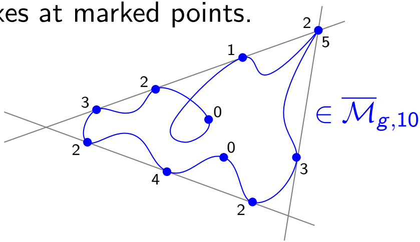
- Each ψ_i is codim-1.
- Defined by imposing local constraints on tangent vector fields.

These intersection numbers do not uniquely determine $[Y]$; even so, along with other classes, they "detect a lot of its shape".

The Double-Double Ramification Cycle

DDR cycle $=$ locus of curves in $\overline{\mathcal{M}}_{g,n}$ that:

- admit a map to $\mathbb{P}^2$ of a given degree;
- has given tangency orders to axes at marked points.



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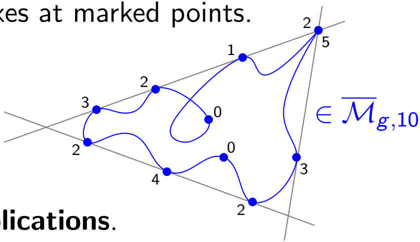

Why is it interesting?

1. Relates to **PDEs**.
2. Downstream **enumerative applications**.
3. The **DR cycle** (curves $C \xrightarrow{d:1} \mathbb{P}^1$ with given zeros/poles) was well-studied (Janda-Pandharipande-Pixton-Zvonkine, 2016).

The Shape of the DDR Cycle

My project: compute several DDR intersection numbers using elementary arguments.

Explicit result for $g = 1$, $a_0 = b_0 = 0$:

$$\int_{\overline{\mathcal{M}}_{1,n+1}} \mathrm{DDR}_1 \psi_0^{n-1} = \frac{1}{24}\left(\sum_{i<j}(a_i b_j - a_j b_i)^2 - \sum_i \gcd(a_i, b_i)\right)$$

Key tool: intersection theory of **toric blowups**.

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(Holmes-Molcho-Pandharipande-Pixton-Schmitt, 2024) Describes how to compute the DDR cycle, via the **logarithmic DR cycle**.

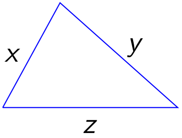
Acknowledgements

Supervisor: Dhruv Ranganathan, Ajith Urundolil Kumaran

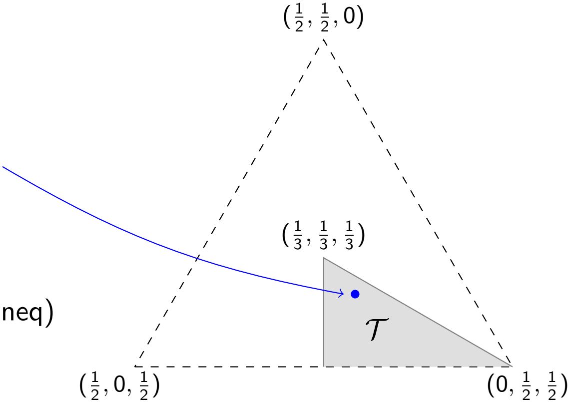
Summer Research in Mathematics (SRIM) scheme

Funding: Trinity College Summer Studentship Scheme

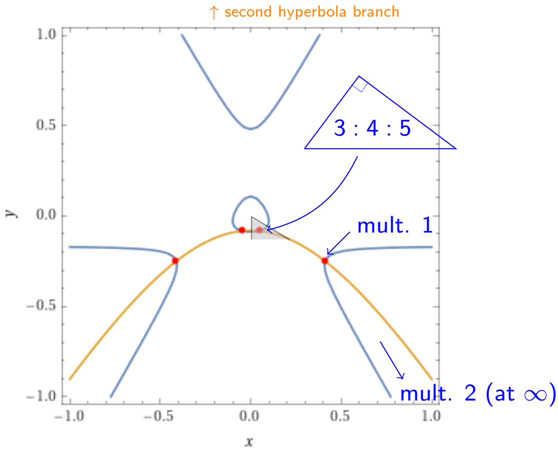
Reference for explicitly building $\mathcal{T}$



- $x + y + z = 1$
- $0 < x \leq y \leq z$
- $x + y > z$ ($\triangle$-ineq)



Reference for Bézout on $\mathcal{T}$



Source: WolframAlpha