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The computation of eigenvalues and eigenvectors is a cornerstone of linear algebra, with profound applications across science and engineering. The power method and the inverse power method are classical iterative algorithms designed to find the largest and the smallest eigenvalue respectively. In recent years, many problems in fields such as machine learning, data science, and image analysis have led to non-linear analogues of eigenvalue problems.

The aim of the project was to generalise the notion of eigenvectors and eigenvalues to the non-linear case in a suitable way, i.e. in such a way that the key properties from the linear case are preserved, and to suggest algorithms – to be more precise, generalisations of the power and the inverse power method – for computing eigenvectors and eigenvalues in the non-linear case. The key result was to prove convergence of this algorithms to stationary points of the generalised Rayleigh quotient using the so-called Kurdyka-Lojasiewiez-property (the KL property).

To provide some background, recall the definition of an eigenvector and eigenvalue of a matrix $A \in \mathbb{R}^{n \times n}$: We say that a vector $u \in \mathbb{R}^n \setminus \{0\}$ is an eigenvector of A iff u satisfies the equation $Au = \lambda u$ for some $\lambda \in \mathbb{R}$. For a symmetric matrix A, the eigenvectors are exactly the stationary points of the Rayleigh quotient

$$R(u) = \frac{\langle Au, u \rangle}{||u||^2}$$

and the extremal eigenvalues, i.e. the largest and the smallest eigenvalue, can be characterised as the maximum and minimum of the Rayleigh quotient.

So one way to generalise the notion of eigenvalues to (non-linear) functions $F, G \in C^1(\mathbb{R}^n)$ is to define the eigenvectors of (F, G) as the stationary points of the generalised Rayleigh quotient,
$R(u) = \frac{F(u)}{G(u)}, where$G ¿ $0, i.e.$u$satisfiesthefollowinggeneralisedeigenvalueproblem$

$$\nabla F(u) = R(u) \nabla G(u).$$

The central theoretical challenge was to prove convergence of our proposed algorithms to stationary points of the generalised Rayleigh quotient, despite the lack of convexity in general. The key assumption enabling this convergence analysis is the KL - property.

The KL property is in some sense a measure of whether the function is amenable to sharpness around a point or not. A function $f : \mathbb{R}^n \to \mathbb{R}$ is sharp around $\bar{x}$ iff there is a neighbourbood U of $\bar{x}$ such that $||\nabla f(x)|| \geq 1$ for all $x \in U \setminus \{\bar{x}\}$. The KL property at $\bar{x}$ means that f can be sharpened around $\bar{x}$ by reparameterizing its values with a function satisfying certain conditions.

This work extends classical methods to a non-linear setting by generalising the Rayleigh quotient and proposing algorithms for computing its stationary points. The KL property plays a central role in the convergence analysis, replacing traditional assumptions with a more flexible and powerful tool for modern non-linear problems.