Quantifying Simplicity For Finitely Generated Groups

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I once held an idealized view of what mathematicians do. I imagined that they mostly observed a phenomenon or concept, and their observations sparked curiosity, eventually leading them to ask the most natural questions. With these questions in mind, the "play" with the problem would begin. At the time, I did not have a clear sense of how this play might unfold; I supposed there were many possible directions, so my imagination of the process grew vague in this stage: Perhaps they created new settings in which to observe the elements of play; perhaps they shifted their perspective to familiar concepts and sought the best possible correspondence; perhaps an unexpected connection emerged between their current question and a curiosity they had explored before. These ideas were, of course, not well grounded, since I had no real experience in mathematical research before the Philippa Fawcett Internship. This internship offered me an amazing opportunity to engage in a research project where I could see how all of these "dramatic" modes of play appear in the work of mathematicians.

There are many interesting maps on the circle to study. A well-known example is the *rotation map*, usually denoted by R_α. The idea is simple: R_α takes each point on the circle S^1 to the point rotated by an angle of α. While this setting is easy to imagine, very intricate behavior appears when α is irrational with respect to the circumference of the circle.

A classical result in dynamical systems states that in this case the orbit of every point is dense in S^1. For example, if we start with a point $a \in S^1$ and apply R_α repeatedly, we will never return exactly to a. However, for any neighborhood $B_\varepsilon(a)$ of arbitrarily small size, the orbit will eventually visit it. From this simple fact, many natural questions follow. How often will the orbit of a return to $B_\varepsilon(a)$? If we record the return times, how large can the gaps be between two consecutive visits? Is it possible to give an upper bound for these gaps?

Let us now place the curious dynamics described above into a new setting. Consider the interval of length α on the circle S^1, starting from the point 1 and extending counter clockwise. Denote this interval by U_α. We observe the dynamics of the rotation map R_α, focusing specifically on whether iterates fall inside U_α. Take an arbitrary point $a \in S^1$ and iterate it under R_α. For each iterate (allowing both positive and negative powers), we record whether it lands in U_α. If the iterate lies in U_α, we assign the symbol 1; otherwise, we assign 0. In this way, the orbit of a under R_α is encoded as a bi-infinite sequence in $\{0,1\}^{\mathbb{Z}}$.

We have now associated a sequence to each point on the circle. One can show that every point corresponds to a unique sequence. If we take all points on the circle and collect their corresponding sequences, what subset of $\{0,1\}^{\mathbb{Z}}$ do we obtain? As it turns out, a very interesting one — let us denote it by X.

An important feature of X is that it is invariant under the shift map. In other words, if we take any sequence in X and shift it one place to the right, the resulting sequence is still in X. This means that the system consisting of the space X, equipped with the shift map, forms a well-defined dynamical system.

Within this new setting — inspired by our original circle dynamics — we encounter another dynamical system, and familiar types of questions naturally arise. We may ask about the different blocks of zeros and ones that occur in members of X. Not every block is possible, of course, but once a block appears in one element, will it appear again? If so, how frequently will it recur? And what are the sizes of the gaps between two consecutive appearances? $(*)$

During our project, we experimented with different ways of approaching these questions. Interestingly, our original motivation was not the dynamical setting just described, but rather a problem in group theory.

Group theorists have long been interested in a particular class of groups known as *simple groups*. These groups enjoy the property that, for any two elements x and g in the group, the element x can be written as a product of conjugates of g. Once again, a natural question arises: can we place a bound on the number of conjugates required? $(**)$

For me, one of the most interesting parts of the project was discovering the unexpected connection between these two settings. The preceding question for a category of simple groups can, in fact, be translated into dynamical inquiries about X. More precisely, one can construct a group based on X, denoted G_X, which is an example of a finitely generated infinite simple group — a rare and intriguing category for group theorists. Answering question $(**)$ for G_X thus translates into addressing the dynamical questions $(*)$ about X.

The main objective of our project, therefore, was to analyze the dynamical properties of X and use them to establish bounds related to question $(*)$ in the setting of the group G_X.

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