Quantifying Simplicity For Finitely Generated Groups

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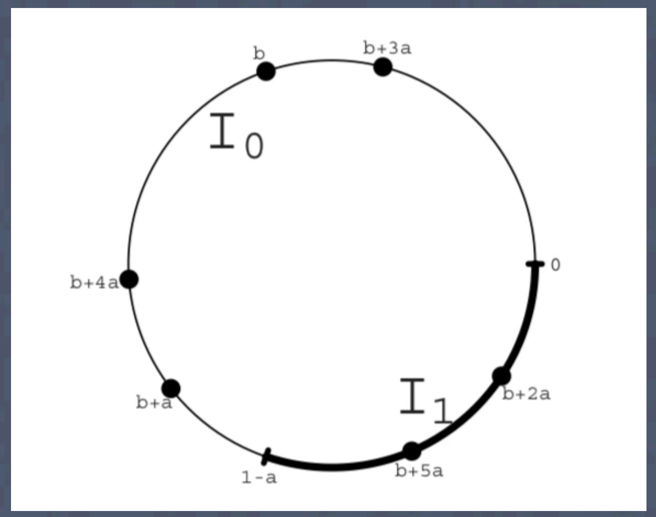
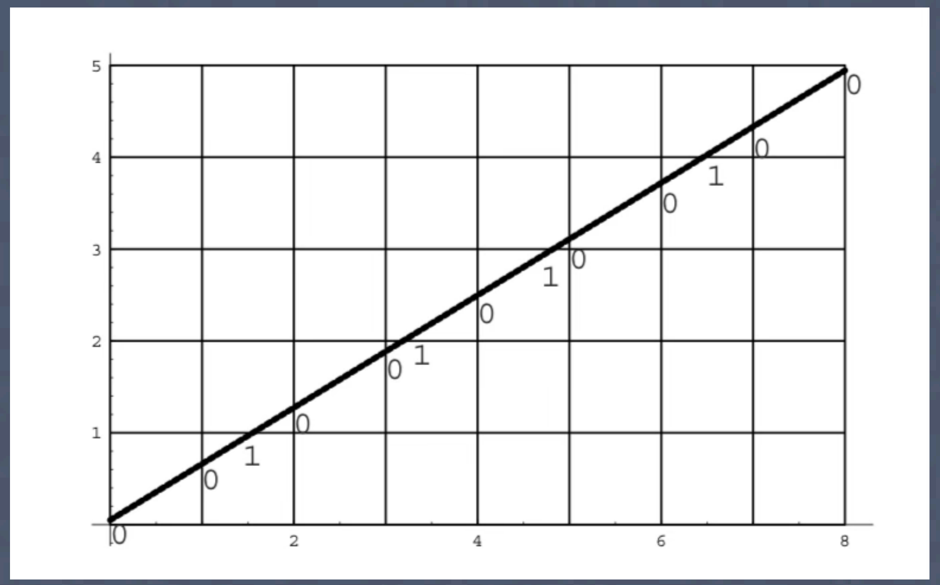
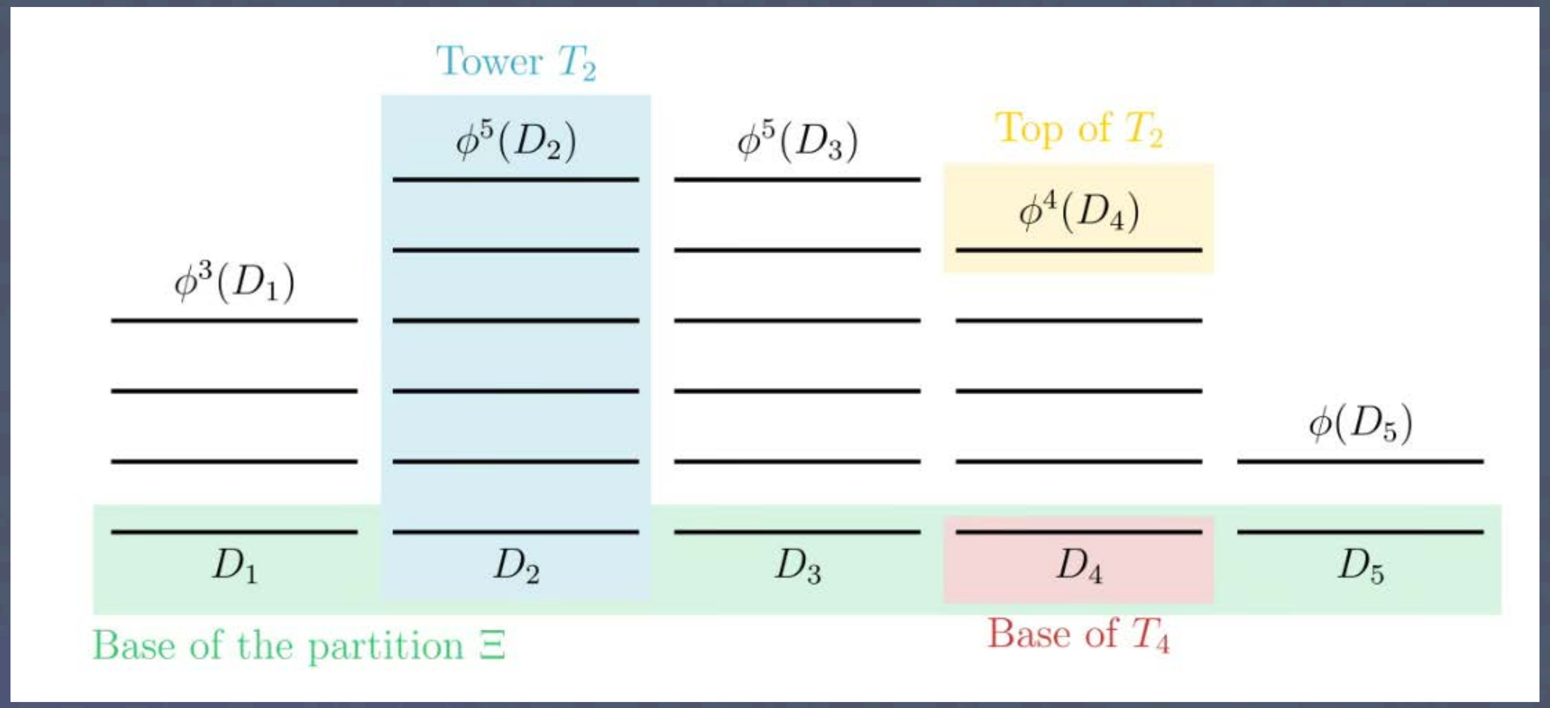
Setup and background

Dynamical systems and group theory!

- The Cantor set and cylinders
- The shift map
- Sturmian sub-shifts
- Dynamical properties
- An infinite finitely generated simple group!







The cantor set

Let A be a finite alphabet set. We think of the cantor set as the set $A^{\mathbb{Z}}$. We equip this set with the topology generated by cylinder sets.

Definition of a cylinder set

A cylinder set of $A^{\mathbb{Z}}$ *is a set of the form:*

$$\langle\langle u_k, \ldots, u_{-1}, u_0, u_1, \ldots, u_l \rangle\rangle = \{y \in A^{\mathbb{Z}} : y_i = u_i \text{ for } k \leq i \leq l\},$$

where $k, l \in \mathbb{Z}$ *with* $k \leq 0 \leq l$ *and* $u_i \in A$ *for* $k \leq i \leq l$. *For* $u \in A^n$ *and* $1 \leq i \leq n$, *we write*

$$\langle\langle u \rangle\rangle_i = \langle\langle u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_n \rangle\rangle.$$

We equip the cantor set with the topology generated By these sets.

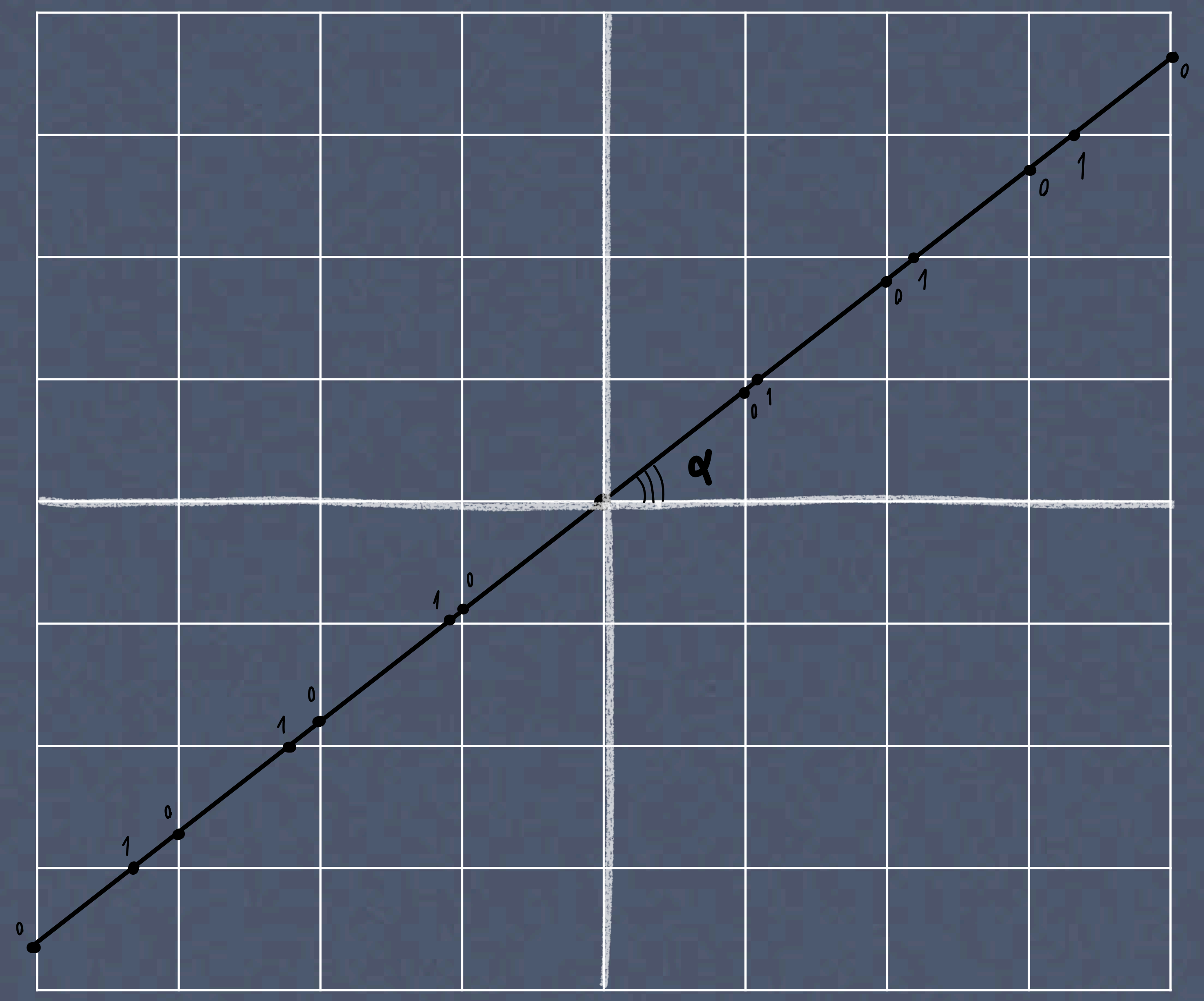
The shift map

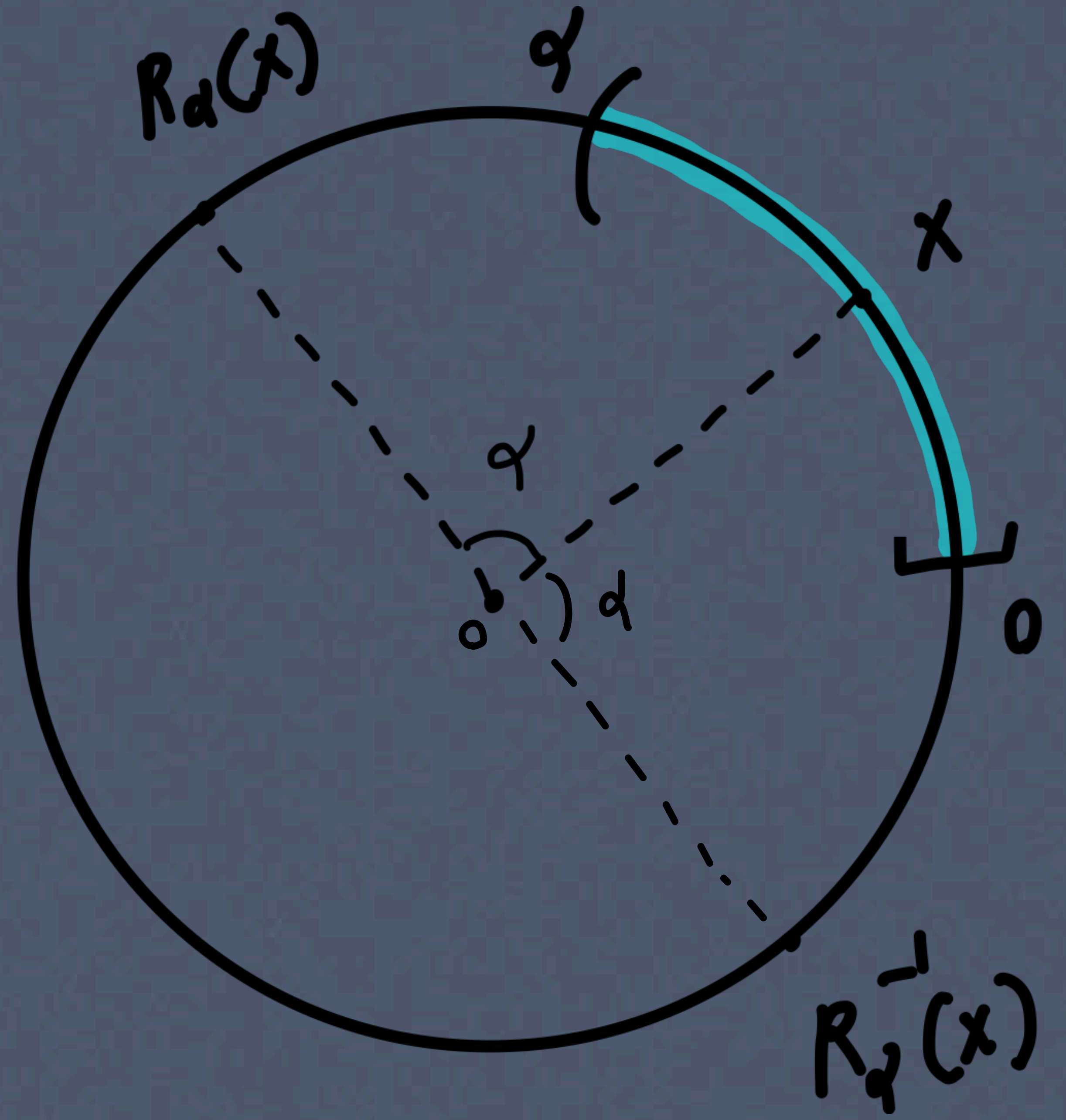
The *shift map* σ on the full shift $A^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose ith coordinate is $y_i = x_{i+1}$.

The operation σ maps the full shift $A^{\mathbb{Z}}$ onto itself:

$$x \;=\; \cdots\; x_{-3}\; x_{-2}\; x_{-1}\; x_0\; x_1\; x_2\; x_3\; \cdots$$

$$y = \sigma(x) \;=\; \cdots\; x_{-2}\; x_{-1}\; x_0\; x_1\; x_2\; x_3\; x_4\; \cdots$$





Take $\alpha \in \mathbb{Q}^c$. Code the intersections of the line with the grid.

Take the orbit of x under the rotation R_α and code it with respect to the interval U_α

$$h : S^1 \longrightarrow \{0,1\}^{\mathbb{Z}}$$

Take h as our coding of the unit circle.

$$X = \overline{h(S^1)}$$

X will be invarient under the shift map.

The dynamical system (X, σ) is called a subshift.

(X, σ) has no periodic points.

Every $x \in X$ has a dense orbit under the shift map.

How often do we visit the cylinders?

How long does it take to come back to a cylinder set?

We can construct an interesting group Γ based on this dynamical system!

A group G is simple iff for every element $x \in G$ and every element $g \in G$, x can be written as a product of conjugates of g.

Γ is a simple group!

The dynamical properties of the sub-shift relate to the properties of this product in Γ.

Thank you!